

IEST5004

9bj]fcba YbhJ``A UbU[Ya YbhF YgYUfW\ *\bhYfbg\]d

HYfa 'H\fYY''##'&\$&\$

7ci fgY'Cj Yfj]Yk

GHJZ:7cbHJWh:8YHJ]`g

7cbj Ybcfg

B Ua Y	9a U]`	5j U]`UV]`] m	@cWUh]cb	D∖cbY
HYa U'A]`ghY]b	hYa U"a]`ghY]b4 i bgk "YXi "Ui	6miUddc]bha Ybh	A cfj Yb [.] 6fck b	
		Xif]b['Wcbgi`HUH]cb	' *('fbf'cb`]bY	
		\ci fg	j]U'HYUa g'Xi Y	
		fK YXbYgXUmgž'‱	hc:X]ghUbWY	
		U"a "`!`%&`d"a "Ł	`YUfb]b['cf	
			7 cj]XŁ	

GWXcc`7cbHJWhi±bZcfa Uh]cb

GWkcc``cZ<iaUb]h]Yg'UbX'@Ub[iU[Yg

@c\\U||cb. G\\cc``C\\Z|\\\žA\cfj\Yb'6fck\b'6i]\X]b[\z'@\j\Y``&z'&),

CdYb]b['<ci fg. 'A cbXUm!': f]XUmž'- Ua '!') da

D\cbY.'Ž*%'&'-',)'%*,%

: UI . Ž*%'&'-',) ', +\$)

9a U]`. \<u>U`4 i bgk "YXi "Ui</u>

7ci fgY'8YW]`g

7 fYX]hDc]bhg**

Gi a a UfmcZh\Y 7ci fgY

H\Y'cVYWIJ Y'cZh\Y'9bj]fcba YbhU``A UbU[Ya Ybh]bhYfbg\]d']g'hc'dfcj]XY'mci 'k]h\'h\Y'cddcfh b]lmhc'k cf_k]h\'Yl dYf]YbWX'Ybj]fcba YbhU``dfUWJh]cbYfg']b'U'dfcZYgg]cbU``gYhJb["'Mci 'k]``i gY'h\Y'_bck `YX[Y [U]bYX']b'dfYj]ci g'A 9A 'Wci fgYgž'UbX'hc'Udd`m]hhc'dfc YWhUg_g'h\fci [\ci hh\Y']bhYfbg\]d"'8YhU]`g'cZh\Y HUg_g'k]``'VY'UffUb[YX']b'Wcbgi `HUh]cb'k]h\'h\Y'Wci fgY'WccfX]bUhcf'cf'a i hi U``mU[fYYX!i dcb'UWUXYa]W ghUZgi dYfj]gcf'UbX'mci f'cb!g]hY'gi dYfj]gcf'fUb'Uddfcj YX'ghUZa Ya VYf'k]h\]b'df]j UhY'cf'di V`]WgYWcf]bgh]h h]cb'cf'cf[Ub]gUh]cbgž'h\Y'Wci fgY'WccfX]bUhcf'k]``k cf_k]h\ mci 'UbX'h\Y'U[YbW]Yg'hc'Ybgi fY'mci 'UfY'Z ``mdfYdUfYX'UbX'h\Uh'h\Y'dfc'YWh]g'Uddfcdf]UhY'"

7ci fgY @YUfb]b['Ci hWca Yg

- %" Ghi XYbhg``YUfb`\ck 'hc`XYj Y`cd`U`dfcZYgg]cbU`#YgYUfVX`dfc^YWhh\fci [\``]U]gcb`k]h\`U`dUfhbYf cf[Ub]gUh]cb
- &" Gh XYbhg'a i ghXYj Y`cd'dfc YWhU]a g'UbX'gWcdY
- '" K]h\n\Y`dUffbYf`cf[Ub]gUf]cbžghi XYbhg`a i ghXYj Y`cd`UbX`Udd`miUddfcdf]UhY`a Yh\cXc`c[]Yg`hc UXXfYgg`h\Y`U]a g`cZh\Y`cf[Ub]gUf]cb`UbX`h\Y`ghi XYbh'dfcYVh
- ("Ghi XYbhg`YUfb'hc'WcbXi Wh`]hYfUhi fY'fYj]Yk g`]b'hc'UXXfYgg'gdYW[Z]WfYgYYUfWX'hcd]Wg"
-) " Gh XYbha UbU`maYžfYdcfhUbX]bhYfdfYhZbX]b[a
- *" Gh XYbhg`YUfb`hc`XY`]j Yf`WcbWi g]cbg`UbX`fYWca a YbXUhJcbg`h\fci [\`fYdcfh]b[`UbX`cfU``XY`]j Yfm

HYUWX]b['GhfUhY[]Yg'

HYUW\]b['glfUhY[]Yg']bWi XY'Wcbgi `HUh]cb'UbX'gi dYfj]g]cb'a YYh]b[g'k]h\'mci f'Wci fgY'Wc!cfX]bUhcf'cf a i hi U``mU[fYYX!i dcb'UWUXYa]WgHUZZgi dYfj]gcfz'U'\UbXg!cb'k cf_]b['fY`Uh]cbg\]d'k]h\'mci f'\cgh cf[Ub]gUh]cbz'UbX'fYdcfh]b['i]U'h\Y'Wci fgY'UggYgga Ybhg"

5ggYgga Ybh

H\Y`Zcfa Uhz``Yb[h\z`UbX`gWcdY`cZU```Ugg][ba Ybhg`]g`Zcf`bY[ch]Uh]cb``VYhk YYb`mci z`mci f`U[fYYX!i dcb]bhYfbg\]d`gi dYfj]gcfz`UbX`mci f`\cghicf[Ub]gUh]cb"`A cfY`[i]XUbW\k]```VY`[jj Yb`]b`Wcbgi `hUh]cb'a YYh]b[g YUf`m]b`h\Y`HYfa `k]h\`mci f`gi dYfj]gcf`UbX`\cghicf[Ub]nUh]cb"

5ggYgga Ybh'HUg_g'

5ggYgga Ybh'hJg_	K Y][\h	8i Y'8UHY	Gh XYbh@YUfb]b[Ci hWca Yg 5ggYggYX
:]bU`F Ydcfh	* \$1	Bch5dd`]WUV`Y	' <u>ž(</u> ž) ž*
Cfu``dfYgYbrUh]cb`cZfYgi`hg UbX`Z bX]b[g`cZ]bhYfbg\]d dfcYVhi	&\$ı	Bch5dd`]WUV`Y) ž*
±bhYfbg\]d`fYgYUfWX`Vf]YZ	%\$ı	Bch5dd`]WUV`Y	%ž&
=bHYfbg\]d 8]Ufmi %\$1		Bch5dd`]WUV`Y	%

5ggYgga Ybh8YHJ]`g

5ggYgga Ybh%:]bU`FYdcfh

GhUfhXUhY. Bch5dd`]WUV`Y

 $8 YHJ]^*g.:]bU`^fYdcfhXYHJ]^]b[^UJa gž'gWcdYž'a Yh\cXgž'UWfJj]h]Yg^WcbXi WhYXž'fYgi ^hg#JbX]b[gž'fY^Yj Ubh ^]hYfUhi fYž'UbX^fYWca a YbXUh]cbg^Zcf^h\Y^ccf[Ub]nUh]cb"$

F Yj]Yk 'Vm'Wci fgY'Wcbj Ybcf'UbX'Wca a Ybhg'Zfca 'cf[Ub]gUh]cb'gi dYfj]gcf"

5XX]h]cbU`'XYhU]`q.

D`YUgY`WcbHJWiU`[ccX!Z]hi9bj]fcba YbhiUbX`GcW]YhmiUWJXYa]WgHJZa Ya VYf`hc`fYei Yghh\YmiWc!gi dYfj]gY mci f`k cf_k]h\`mci f`\cghcf[Ub]gUh]cb"

Gi Va]gg]cb bch/g.: i ```XYhU]`g`hc`VY`bY[ch]Uh/X`k]h\`\cgh'cf[Ub]gUh]cb`UbX`]bh/fbg\]d`gi dYfj]gcf

 $\label{thm:limit} \begin{tabular}{ll} Hi & fb & fb & gYh &$

5ggYgga Ybh& CfU`dfYgYbhJhcbcZfYgi`hg UbXZ]bX]b[gcZ]bhYfbg\]ddfcYWh

GhUfhXUhY. Bch5dd`]WUV`Y

 $@\textbf{Yb[h}. : i ```XYHJ]`g`hc`VY`bY[ch]UhYX`k]h. `\cghcf[Ub]gUh]cb`UbX`]bhYfbg\]d`gi dYfj]gcf$

 $8YHU]`g. C fU``dfYgYbHUH]cb``cZZ]bX]b[g`hck UfXg`YbX`cZ]bHYfbg\]d \\ DUffbYf`UbX`Wcbj Ybcf`Wca a YbhUbX`h\]g`dfcj]XYg`Ugg]gHUbW'`k]h\`Z]bU``fYdcfhWca d`Yh]cb$

5XX]h]cbU`XYhJ]`g.

H\Y`Zcfa Uh'UbX`WcbhYbhk]```VY`XYhYfa]bYX`Vmih\Y`]bhYfbg\]d`gi dYfj]gcf`UbX`h\Y`\cghicf[Ub]gUh]cb"

Gi Va]gg]cb bch/g.: i ```XYhU]`g`hc`VY`bY[ch]Uh/X`k]h\`\cghicf[Ub]gUh]cb`UbX`]bh/fbg\]d`gi dYfj]gcf

Hi fb]hjb gYhjb[. H\]g]g bchU Hi fb]hjb Ugg][ba Ybh

5ggYgga Ybh' . *bhYfbg\]d fYgYUfW Vf]YZ

GhUfhXUhY. Bch5dd`]WUV`Y

8YHJ]`g.G\cfhVf]YZk]h\`dfc^YWiU]a gžigWcdY`UbX``YHYf`cZU[fYYaYbhik]h\`cf[Ub]gUh]cb`]bWiXYX" FYi]Yk`VmWcifqY`WcbjYbcf

5XX]h]cbU`XYhJ]`g.

H\Y`Vf]YZZcf`h\Y`\bhYfbg\]d`]g`YghUV`]g\YX`]b`Wcbgi``hUh]cb`VYhk YYb`h\Y`]bhYfbg\]d`gi dYfj]gcf`UbX`h\Y\cghcf[Ub]gUh]cb"`Bcfa U``mh\Y`Vf]YZk]```VY`XYhYfa]bYX`UbX`k f]hhYb`k]h\]b`h\Y`Z]fghihk c`k YY_g`cZh\Y HYfa"

Gi Va]gg]cb'bch/g.: i ```XYhU]`g'hc'VY'bY[ch]Uh/X'k]h\ \cghicf[Ub]gUh]cb'UbX']bh/fbg\]d'gi dYfj]gcf

Hi fb]hjb gYhjb[. H\]g]g bchU Hi fb]hjb Ugg][ba Ybh

5ggYgga Ybh(. +bhYfbg\]d 8]Ufm

GhUfhXUhY. Bch5dd`]WUV`Y

8 YHJ]`g. ♣ `cfXYf`hc`Ugg]ghk]h\ `ghi XYbhdfc YWha UbU[Ya Ybhg_]``gžghi XYbhg`a i gh_YYd`UbX`gi Va]hU`X]Ufm fYdcfh]bX]WUh]b[`UWNjj]h]Ygž'dfc YWhd`Ubb]b[`UbX`h]a Y`]bYg" 5 ggYggYX`Vmh\Y`Wcbj Ybcf

Hi fb]hjb gYhjb[. H\]g]g bchU Hi fb]hjb Ugg][ba Ybh

5 https://bxubwww.Fyei]fYaYbhg

 $\hbox{Ghi XYbhg'UfY'ghfcb[`m'YbWc'i fU[YX'hc'UhhYbX'U```WUggYg'UbX'fYj]Yk'``YWhfY'fY'WcfX]b[g''' } \\$

7ci fgY'GWXYXi `Y

J]Yk 'WUgg'hja YHUV'Y

H]a YtUV`Y

8UhY	HmdY	7 cbh/bh
K YY_'%'%('GYdhYa VYf'! % 'GYdhYa VYf		5 k YY_`mgW.YXi `Y`]gʻbchdfcj]XYX`\YfY"'8i f]b[`cf VYZcfY`h\Y`Z]fghk YY_cZHYfa žh\Y`gWcdY`cZh\Y]bhYfbg\]dʻbYYXgʻhc`VY`XYhYfa]bYX`]b`bY[ch]Uh]cbg VYhk YYb`\cghcf[Ub]gUh]cbžmci ž'UbX`mci f`U[fYYX! i dcb`UWUXYa]WghUZZ]bhYfbg\]dʻgi dYfj]gcf"`5```ch\Yf gW.YXi `]b[`UbX`fYei]fYa YbhgʻZck `Zfca `h\]g`YUf`m YghUV`]g\a YbhcZh\Y`]bhYfbg\]d"`

FYgci fWyg

DfYgWf]VYX F Ygci fWYg

H\Y`fYgci fWrg`bYYXYX`Zcf`mci f`\bhYfbg\]d`k]```VY`XYhYfa]bYX`Vmh\Y`gWcdY`UbX`]bhYbhcZh\Y`dfcYWb`UbX`Vmh\Y`fYei]fYaYbhg`cZmci f`\cghicf[Ub]gUh]cb`UbX`mci f`U[fYYX!i dcb`UWUXYa]WgHUZZ]bhYfbg\]d gi dYfj]gcfž'fY`UhYX`hc`mci f`hcd]W

FYWca a YbXYX FYgci fW/g

 $\label{thm:continuity} $$Mci 'UfY'YI dYWMYX'hc'YI YfW]gY'U'\][\XY[fYY'cZ]b]h]Uh]j Y']b'Z]bX]b['mci f']bhYfbg\]d'UbX'fYgYUfWX'a UhYf]U'g fY'Yj UbhZcf'mci f'dfc'YWf'$

7ci fgY 9j U'i Uhjcb UbX 8Yj Y'cda Ybh

@_Y'U``A 9A 'Wci fgYgžh\Y'UddfcUW\'hc'h\Y'+bhYfbg\]d']g'XYj Y`cdYX'cb'h\Y'VUg]g'h\Uh]bXYdYbXYbhUbX Udd`]YX'`YUfb]b[']g'\][\`mj U'i YX'Vmgh XYbhgž'VUgYX'cb'dUghYl dYf]YbW\'UbX'dfYj]ci g'ZYYXVUW_" Gh XYbhg`\Uj Y'U`gc'Yl dfYggYX'ghfcb[']bhYfYgh]b'h\Y'Yl hYfbU` Wca a i b]m#[cj Yfba Ybh#B; C#bXi ghfmYl dYf]YbW\'h\Uh\\Y'+bhYfbg\]d'cZYfg"

Gi Va]gg]cb cZ5ggYgga YbhHUg_g

Hi fb]h]b Gi Va]gg]cb

#mci 'YbWci bhYf'U'dfcV'Ya 'k \Yb'UthYa dhJb['tc'gi Va]hmci f'Ugg][ba Ybht\fci [\'Hi fb]hJbž'd'YUgY
hY'Yd\cbY'9| hYfbU'Gi ddcfhcb'-',) '''' %cf'Ya U]''h\Ya 'cb'Yl hYfbU'hY'hgi ddcfh4 i bgk "YXi "Ui '"'Gi ddcfh
\ci fg'UfY', .\$\$Ua '-'%\$.\$\$da 'cb'k YY_XUmg'UbX'-.\$\$Ua '-').\$\$da 'cb'k YY_YbXg'fl *) 'XUmg'U'mYUft\"'#c'i'#c'i'
UfY'i bUV'Y'hc'gi Va]hmci f'Ugg][ba YbhXi Y'hc'U'ZUi 'hk]h\ 'Hi fb]hJb'mci 'a Um'Udd'm\Zcf'Ub'Yl hYbg]cb\ZVi hmci
a i ghfYhU]b'mci f'hJ\W_Yhbi a VYf'\Zfca '9| hYfbU''Gi ddcfhfU'cb['k]h\ 'Ubmch\Yf'fY'Yj UbhXcW a Ybhgthc
]bWi XY'Ug'Yj]XYbW''hc'gi ddcfhmci f'Yl hYbg]cb'Udd']\WUh]cb'''#mci 'Ya U]''9| hYfbU''Gi ddcfhmci 'k]''

5 WUXYa]W<cbYghmUbX D`U[]Uf]ga

D`U[]Uf]ga i]gii g]b[ih\Yik cfXgicfi]XYUgicZch\YfgiUbXidfYgYbh]b[ih\Ya iUgimci fick b"i=iWUbihU_Yia Ubm Zcfa gžZfca iXYi]VYfUhYiWXYUh]b[ihciUWW]XYbhU`mWcdm]b[iZfca iUigci fWYik]h\ci hiUW_bck iYX[Ya Ybh'i

- i bXYfgHUbX'mci f'f][\hg'UbX'fYgdcbg]V]`]h]Yg'Ug'U'gh XYbhUhI BGK
 VY'Uk UfY'cZd`U[]Uf]ga ž'Wcdmf][\hz'I BGK 'Gh XYbh7cXY'cZ7cbXi WhUbX'5VWWdhUV`Y'I gY'cZ IBGK =7 HF Ygci fWrg Dc`]Wh

5 WUXYa]W = b Zcfa Uh cb

: cf'YggYbhJU`'gh XYbh']bZcfa UhJcb'fY`UhJb['hc.

- fYei Yghg 'Zcf'Yl hYbg]cb/
- `UhY`gi Va]gg]cbg`[i]XY`]bYg/
- fYj]Yk 'cZa Uf_g/
- I BGK '< YU'h\ 'UbX'GUZYhm'dc`]\\\Yg/
- YI Ua]bUh]cb dfcWYXi fYg/
- gdYWJU`Wcbg]XYfUHjcb`]b`H\Y`Yj YbhcZ]``bYgg`cf`a]gUXj Ybh fY/
- gh XYbhYei]mUbX X]gUV]`]m/
- UbX'ch\Yf'YggYbhJU`'UWUXYa]W]bZcfa UhJcbž'gYY

\hdg.#k k k "Ufhg"i bgk "YXi "Ui #W ffYbhgh XYbhg#UWJXYa]W]bZcfa Uh]cb#dfchcWc`g![i]XY`]bYg#

⇒a U[Y'7fYX]h

D\chc Vm8Ub]Y`FcV]bqcb

7F = 7CG

7F = 7CG Dfc|]XYf 7cXY. \$\$\$-,;

5W_bck `YX[Ya YbhcZ7ci bhfm

 $K Y : UW\underline{b} ck `YX[Y : h, Y : 6YXY[U : dY cd `Y : k \ c : UfY : h, Y : hfUX]h\underline{b} cb : `W ghcX]Ubg : cZh, Y ``UbXg : cb : k \]WX : I B G K ? Ybg]b[hcb : WUa di g :]g ``cWUhYX''$