

# **ARTS3633**

DfcZYgg]cbU``>UdUbYgY'6

GYa YghYf'Hk c''#'&\$%

## 7ci fgY<sup>-</sup>Cj Yfj ]Yk

## 'GhJZZ'7cbhJWh'8YhJ]`g'

## 7cbj Ybcfg

B Ua Y	9a U]`	5j U]`UV]`] <b>m</b>	@cWUh]cb	D\cbY
M a ]_c '< Ug\]a chc	m"\Ug\]a chc4 i bgk "YXi "Ui	Hi Y <sup>.</sup> %\$.\$\$!%&.\$\$žcf	A 6&\$'	\$&!-',)!'+&
		VmUddc]bha Ybh		&

## GWXcc``7cbHJWhʻ±bZcfa Uh]cb

GWkcc``cZ<iaUb]hJYg'UbX'@Ub[iU[Yg

@c\\Ulrighthgap. G\\\cc``C\Z]\\\\ž\A\cfj\Yb`6fck\b'6i]\\]b[\ž\@Yj\Y``&\z\&),

CdYb]b[ '<ci fg. 'A cbXUm!': f]XUmž'-Ua '!') da

D\cbY. 'Ž\*%'&'-',)'%\*,%

: UI . Ž\*%'&'-' , ) ', +\$)

9a U]`. \<u>\U`4 i bgk "YXi "Ui</u>

: cf`YggYbhJU``ghi XYbh`]bZcfa UhJcb`fY`UhJb[ 'hc. 'fYei Yghg`Zcf`Yl hYbg]cb/fYj ]Yk 'cZa Uf\_g/cWW dUhJcbU``\YU`h\UbX`gUZYhm'Yl Ua ]bUhJcb'dfcWYXi fYg/gdYWJU``Wcbg]XYfUhJcb`]b'h\Y'Yj YbhcZ]``bYgg`cf`a ]gUXj Ybhi fY/ghi XYbhYei ]hmUbX`X]gUV]`]hm'UbX`ch\Yf`YggYbhJU``UWUXYa ]W]bZcfa UhJcbž gYY`\hdg.#k k k "Ufhg"i bgk "YXi "Ui #W ffYbhghi XYbhg#JWUXYa ]W]bZcfa UhJcb#dfchcWc`g![i ]XY`]bYg#

## 7ci fgY 8YHJ] g

7 fYX]hDc]bhg \* \*

Gi a a UfmcZh\Y'7ci fgY

Gi VYWn5fYU. <SbS` WeWEfgV[We

H\Y'Wei fgY'\Ug'h\fYY'd\UgYgžk\]W. fY\UhY'hc'U'fYgYUfW. dfc YWhcb'h\Y'>UdUbYgY'`Ub[i U[Y']b'cfXYf'hc XYj Y`cd'mci f'`]b[i ]gh]W\_bck `YX[Y'UbX'g\_]``g']b'XYdh\"'=b'h\Y'Z]fghd\UgYž'h\Y'Wei fgY'`YUXg'mci 'hc'Yl Ua ]bY UbX'X]gW gg fYUX]b[g'UbX'j ]gi U``a UhYf]U`g'dfYgYbh]b[ij Uf]ci g'UgdYWg'cZh\Y'W\UfUWYf]gh]Wg'cZh\Y'\Ub[i U[Y''Nci 'fYZ]bY'UbX'Yl dUbX'mci f'\_bck `YX[Y'cZ[fUa a Ufžj cWUVi `UfmžYl dfYgg]cbgž] S`\[žUbX X]ZYfYbhghm`Yg'UbX'fY[]ghYfg']b'>UdUbYgY'h\fci [\`]bhYfUWf]YY'Yl YfW]gYg']b'Vch\'`YWi fYg'UbX'h hcf]U`g" H\YgY'dfUWf]Wg'g'dfcj ]XY'mci 'gca Y'\]bhg'UbX']XYUg'hc'cdhZcf'mci f'fYgYUfW\'hcd]W"+b'h\Y'gYWcbX'd\UgYž mci 'dfYdUfY'U'dfc YWhVmYl d'cf]b['U'gdYW]Z]WUfYU'cZ>UdUbYgY'mci 'Z]bX']bhYfYgh]b['cf'i gYZ\'''5ZhYf dfUWf]W]b['Wa a i b]Wh]cb'gffUhY[]Yg'UbX'g\_]`gž'U'gYf]Yg'cZ]bhYf] ]Yk 'gYgg]cbg'Zcf'XUhU'Wc`YWf]cb'[]j Y'mci cddcfhi b]h]Yg'hc']bhYfUWfik ]h\'mci f'dYYfg'UbX'bUh]j Y'>UdUbYgY'gdYU\_Yfg''+b'h\Y'Z]bU''d\UgYžmci 'k f]hY'U dUdYf'UbX'dfYgYbh]hi]b'>UdUbYgY''Mci 'i gY'mci f'UbU'mh]WD''UbX'Wf]h]WD''h\]b\_]b['Ug'k Y``'Ug'dfYgYbhUh]cb UbX'UWJXYa ]Wk f]h]b['g\_]``g''+h\]g'Wci fgY']g'hui [\hi]b'>UdUbYgY''

#### 5 hh Y WcbWi g]cb cZh ]g Wci fgY h Y gh XYbhk ] ` VY UV Y hc

- %" 9a d`cm\_bck `YX[Y`cZWXUfUWYf]ghWq`cZ>UdUbYgY``Ub[i U[Y"
- &" =XYbhZmck b fYgYUfW ]bhYfYgh]b >UdUbYgY `Ub[i U[Y"
- '" I gY`Wcaaib]WUh]cb`g\_]``g`hc`]bhYfUWnk]h\`U`k]XY`fUb[Y`cZ>UdUbYgY`gdYU\_Yfgž'Zfca`VY[]bbYfg`hcbUh]jY`gdYU\_Yfg"
- ("8YacbglfUhY WcbZXYbWY UbX g\_]``g`]b`cfU``dfYgYbfUh]cb`]b`>UdUbYgY"
- ) "K f]hY hY l hg ]b 'UWUXYa ]W>UdUbYgY"

### HYUW ]b[ 'GhfUhY[ ]Yg'

H\Y'Wti fgY'Ya d`cmg`a i `h]d`Y'hYUW\]b[ `a Yh\cXg`UbX'a cXYg`cZ]bghfi Wf]cb`gi W\`Ugʻj ]gi Uʻž'Ui X]hcfmUbX \_]bUYgh\Yh]WVUgYX`a UhYf]Uʻg`UbX'YI YfW]gYg"'\do ``YWfi fYgž'mci ``YUfb`UVci hih\Y'W\UfUWYf]gh]Wg`cZh\Y >UdUbYgY``Ub[i U[Y'VmYI Ua]b]b[ `a UhYf]UʻgʻfY`UhYX'hc YUW\ k YY\_fig`hcd]Wf\do hi hcf]Uʻgž'mci `dfUWf]WY`hc`i gYh\Y]f`>UdUbYgY``]b[ i ]gh]W\_bck `YX[Y'hc'i bXYfghUbX'h\Y'hcd]Wg`]b!XYdh\ k ]h\ 'Wta dfY\Ybg]j Y'hUg\_gž'UbX YI dfYgg`mci f'j ]Yk `]b`X]gWi gg]cbg"'H\fci [\'h\Y'fYgYUfW\'dfcYWfimci 'XYj Y`cd'h\Y'g\_]``g`cZfYZYfYbW\'fYUX]b[ ž`]bhYfj ]Yk ]b[ žXUhU`UbU`mg]gž'cfU``dfYgYbhUhJcb`UbX`U\WUXYa ]Wk f]h]b[ `]b`>UdUbYgY'': cf`Ub ]bX]j ]Xi U``dfcYWfimci 'UfY']bj c'j YX']b'j Uf]ci g`UWfjj ]h]Yg`VYmcbX'U`WUggfcca 'hc`\Uj Y`cddcfhi b]h]Yg`hc ]bhYfUWfik ]h\ 'hi hcfg`UbX'ghi XYbhg`Zfca 'X]ZZYfYbh`Yj Y`g`cZ>UdUbYgY``Ub[i U[Y'Wti fgYg`k ]h\]b'h\Y'Ybh]fY >UdUbYgY'dfc[fUa"

## 5ggYgga Ybh

#### 5ggYgga YbhHUg\_g

5ggYgga Ybh'hUg_	K Y][ \h	8iY'8UHY	Gh XYbh@YUfb]b[ Ci hWca Ygʻ5ggYggYX
K f]thYb HYgh	&) ı	\$(#\$-#&\$% `%\$.\$\$`5A	%Ž)
F YgYUfWX Dfc YWfi	) \$1	Bch5dd`]WUV`Y	%ž&Ž Ž( ž)
GdYWJU``Dfc^YWi	&) ı	&*#%\$#&\$% `\$(.\$\$`DA	%₹ ž(

#### 5ggYgga Ybh8YHJ]`g

5ggYgga Ybh% K f]hYb HYgh

GhUfhiXUhY. \$ (#\$-#&\$% \$-.\$\$ 5 A

**8YHJ]`g**. `Ghi XYbhg`Wca d`YhY`U'hYghifWJ"`%\ci fE`]bWi X]b[`g\cfh`UbX``cb[Yf`Ubgk Yf`ei Ygh]cbg"`Ghi XYbhg fYWY]j Y`U'a Uf\_`UbX`WUb`Wcbgi `hih\Y``YWhi fYf`Zcf`Z fh\Yf`ZYYXVUW\_"

Hi fb]hjb gYhjb[. H\]g ]g bchU Hi fb]hjb Ugg][ ba Ybh

5qqYqqa Ybh& FYqYUfW Dfc YWh

GhUfhiXUhY. 8-#\$, #&\$% %.\$\$DA

**8YHJ**]`g. `Ghi XYbhg`dfYdUfY`Zcf`Ub`UVghfUWh`UbX`\UbXci hg`Zcf`%\$ı ž'dfYgYbh'U'dUdYf`Zcf`&\$ı ``f\WJ\"%)
a ]bi hYgŁž'UbX`k f]hY`U'dUdYf`Zcf`&\$ı ```f\WJ\"&\$\$\$! 'JŁ\": YYXVU\W\_`j ]U`]bX]j ]Xi U``Wca a Ybhgž'fi Vf]\W\*UbX`]b!
WUgg`X]gW gg]cb\"H\]g`]g`h\Y`Z]bU``UggYgga Ybh'HJg\_`Zcf`UhhYbXUbV\\' di fdcgYg\"

Gi Va ]gg]cb bch/g.8 Uh/g Zcf df/g/bh/hjcb UbX dUdYf gi Va ]gg]cb UfY hc VY XYh/fa ]bYX"

Hi fb]hjb gYhjb[. H\]g ]g bchU Hi fb]hjb Ugg][ ba Ybh

5ggYgga Ybh' . GdYWJU` Dfc YWh

GhUfh'XUhY. K YY\_&

@Yb[h\. j Uf]ci g

 $\label{thm:limit} \begin{tabular}{ll} Hi fb]h]b gYhh]b[."H\]g"ugg][ba Ybh]g"gi Va]hhYX"h\fci [\"Hi fb]h]b "UbX"gh XYbhg"Xc"bchgYY"Hi fb]h]b g]a]"uf]hmfYdcfhg" \\ \end{tabular}$ 

#### Gi Va ]gg]cb cZ5ggYgga YbhHUg\_g

Gh XYbhg UfY Yl dYWYX hc di hh\Y]f bUa Yg UbX gh XYbhbi a VYfg cb Yj YfmdU[Y cZh\Y]f Ugg][ ba Ybhg"

#### Hi fb]h]b Gi Va ]gg]cb

#mci YbWci bhYf'U'dfcV'Ya k \Yb'UhYa dhJo['hc'gi Va ]hmci f'Ugg][ ba Ybhh\fci [\'Hi fb]h]bž'd'YUgY
hY'Yd\cbY'9| hYfbU'Gi ddcfhcb'-', ) "''' %cf'Ya U]'h\Ya 'cb'<u>Yl hYfbU'hY'hgi ddcfh4 i bgk "YXi "Ui"</u>"'Gi ddcfh
\ci fg'UfY', .\$\$Ua '-'%\$.\$\$da 'cb'k YY\_XUmg'UbX'-.\$\$Ua '-').\$\$da 'cb'k YY\_YbXg'fl \*) 'XUmg'U'mYUft'''#mci
UfY'i bUV'Y'hc'gi Va ]hmci f'Ugg][ ba YbhXi Y'hc'U'ZJi 'hk ]h\'Hi fb]h]b'mci 'a Um'Udd'mZcf'Ub'Yl hYbg]cbž'Vi hmci
a i ghfYhJ]b'mci f'h]W\_Yhbi a VYf'Zfca '9| hYfbU''Gi ddcfhfU'cb['k]h\'Ubmch\Yf'fY'Yj UbhXcW'a Ybhgt'hc
]bWi XY'Ug'Yj ]XYbWY'hc'gi ddcfhmci f'Yl hYbg]cb'Udd']WUh]cb'''#mci 'Ya U]''9| hYfbU''Gi ddcfhmci 'k]'`
Ui hca UhJWJ'`mfYWY']j Y'U'HJW\_Yhbi a VYfž'Vi h]Zmci 'hY'Yd\cbY'mci 'k]'''bYYX'hc'gdYWJZJWJ'`mUg\_'Zcf'cbY''
Hi fb]h]b'U'gc'dfcj ]XYg'i dXUhYg'cb'h\Y]f'gmghYa 'gfUh' g'cb'Hk]hYf"

; YbYfU`mž'UggYgga YbhitUg\_g'a i ghVY'gi Va ]hhYX'Y`YWfcb]WU``mj ]U'Y]h\Yf'Hi fb]h]b'cf'U'A ccX`Y Ugg][ ba Ybh''=b']bghUbWfg'k \YfY'h\]g']g'bchdcgg]V`Yž']hk ]```VY'ghUhYX'cb'mci f'Wci fgY'g'A ccX`Y'g]hY'k ]h\U`hYfbUh]j Y'gi Va ]gg]cb'XYhU]`g"

#### @UhY 5ggYgga Ybh'DYbU`h]Yg

Gh XYbhgʻUfYʻfYgdcbg]V`YʻZcfʻh\Yʻgi Va ]gg]cbʻcZUggYgga YbhhUg\_gʻVmh\YʻfYei ]fYX`XUhYgʻUbXʻhJa Yg" 8YdYbX]b[ˈcZh\Y`YI hYbhcZXY`Um]bʻh\Yʻgi Va ]gg]cbʻcZUbʻUggYgga YbhhUg\_ʻdUghh\Y`Xi Y`XUhY`UbXʻhJa Yž

- K cf\_'gi Va ]hhyx'%5'hc'% 'XUmg'UZhYf'h\Y'gh]di `Uhyx'XYUX`]bY'k ]``'VY'UggYggYX'UbX'ZYYXVUW\_dfcj ]XYX'Vi h'U'a Uf\_'cZnYfc'k ]``'VY'fYWcfXYX'''=Zh\Y'k cf\_'k ci `X'\Uj Y'fYWf]j YX'U'dUgg'a Uf\_'Vi h'Zcf h\Y'`'UhYbYgg'UbX'h\Y'k cf\_']g'U'Wca di `gcfm'Wci fgY'Wca dcbYbh'f\i fX`Y'fYei ]fYa Ybhz'U'gh XYbh'k ]`` VY'XYYA YX'hc'\Uj Y'a Yhh\Uh'fYei ]fYa Ybh/
- K cf\_'gi Va ]hhYX`&\$`cf`a cfY`XUmg`UZhYf`h\Y`gh]di `UhYX`XYUX`]bY`k ]```bchVY`UWWdhYX`Zcf UggYgga YbhUbX`k ]```fYWY]j Y`bc`ZYYXVUW\_ža Uf\_'cf`[ fUXY"'=Zh\Y`UggYgga YbhHUg\_']g`U`Wca di `gcfm Wca dcbYbhcZh\Y`Wci fgY`U`gh XYbhk ]```fYWY]j Y`Ub`l bgUh]gZJWcfm: U]``fl : Ł'[ fUXY`Ug`U`fYgi `hcZ i bgUh]gZJWcfmdYfZcfa UbWY`]b`Ub`YggYbh]U``Wca dcbYbhcZh\Y`Wci fgY"

H\]g']bZcfa Uh]cb']g'U`gc'Uj U]`UV`Y'Uh

\hdg.#k k k "Uffg"i bgk "YXi "Ui #W ffYbhgh XYbhg#UWJXYa ]W]bZcfa Uhcb#dfchcWc`g![i ]XY`]bYg#

#### GdYWJU`7cbg]XYfUhJcb5dd`JWUhJcbg

 $\label{thm:condition} $$Mci `WUb`Udd`m'Zcf'gdYW]U``Wcbg]XYfUh]cb`k \Yb`]``bYgg`cf`ch\Yf'W]fW a ghUbWfg`]bhYfZYfY`k ]h\`mci f UggYgga Ybh'dYfZcfa UbWf''$ 

G]W\_bYggža ]gUXj Ybhi fY'cf'ch\Yf'W]fW a ghUbWrg'VYmcbX'mci f'Wcblfc``a Um

ł DfYj Ybhmci Zfca Wca d`Yhjb[ 'U'Wci fgY'fYei ]fYa Ybhž

ł "? YYd 'mci 'Zfca 'UthYbX]b[ 'Ub 'UggYggUV'Y 'UWhj ]lmž

ł Ghod moi igi Va ]hh]b[ iUggYggUV Y k cf\_ Zcf U Wci fgYž

ł G][ b]Z]WUbhmUZZYWimci fidYfZcfa UbWijbiUggYggUV`Yik cf\_žVYi]hUiZcfa U``YbX!cZlgYa YghYfiYl Ua ]bUh]cbž'U WUqqihYqhž'Ui`UVcfUhcfmhYqhž'UiqYa ]bUfidfYqYbhUh]cbicfiUbmch\YfiZcfa icZiUqqYqga Ybh'

cfilia fth Yfixyfuljgi]bify`uhjcbihciGdYWju`i7 cbg]XYfuhjcbi]bWi X]b[iiK \Ybihci5dd`mižii<ckihci5dd`mi`ubXiiGiddcfhjb[i8cWia Ybhuhjcbiid`YugYifyZyfihcih\YiGdYWju`i7 cbg]XYfuhjcbik YVg]hY.ii\hdg.#ghiXYbhibgk "YXii"ui #gdYWju`iWcbg]XYfuhjcb

#### 5 WUXYa ]W<cbYghmUbX D`U[ ]Uf]ga

D`U[]Uf]ga `]g`i g]b[ 'h\Y`k cfXg`cf`]XYUg`cZch\Yfg`UbX`dfYgYbh]b[ 'h\Ya `Ug`mci f`ck b"'\lefi\Wb'hU\_Y`a Ubm Zcfa gžZfca `XY`]VYfUhY`W\YUh]b[ 'hc`UVW]XYbhU`mWcdm]b[ 'Zfca `U`gci f\W' k ]h\ci hUW\_bck `YX[ Ya Ybh'

IBGK [fcidgdu[]Uf]ga ]bhch\YZc``ck]b[ WUhY[cf]Yg.

7 cdm]b[.'i g]b['h\Y'gUa Y'cf'j Yfmg]a]`Uf'k cfXg'hc'h\Y'cf][]bU`hYl h'cf']XYU'k]h\ci h'UW\_bck `YX[]b['h\Y gci fW'cf'i g]b['ei chUh]cb'a Uf\_g"'H\]g'U`gc'Udd`]Yg'hc']a U[Ygž'Ufh'UbX'XYg][b'dfcYWhgž'Ug'k Y``'Ug dfYgYbhUh]cbg'k \YfY'gca YcbY'dfYgYbhg'Ubch\Yf'g']XYUg'cf'k cfXg'k]h\ci h'WfYX]h'

⇒ Uddfcdf]UhY 'dUfUd\fUg]b[...7\Ub[]b[ 'U'ZYk 'k cfXg'UbX'd\fUgYg'k \]'Y 'a cghmfYhJ]b]b[ 'h\Y'cf][]bU` ghfi Wh fY 'UbX#cf'dfc[fYgg]cb'cZ]XYUg'cZh\Y'cf][]bU'žUbX']bZcfa Uh]cb'k ]h\ci hUW\_bck `YX[Ya Ybh'H\]g'U`gc Udd`]Yg']b'dfYgYbhUh]cbg'k \YfY'gca YcbY'dUfUd\fUgYg'Ubch\Yf'g']XYUg'cf'k cfXg'k ]h\ci hWfYX]hUbX'hc d]YW]b[ 'hc[Yh\Yf'ei chYg'UbX'dUfUd\fUgYg']bhc'U'bYk 'k \c`Yžk]h\ci hUddfcdf]UhY'fYZYfYbW]b[ "

 $7c^{i}$  g]cb. 'k cf\_]b[ 'k ]th. 'cth. Yfg'Vi h'dUgg]b[ 'c\Zth. Y'k cf\_'Ug'U'dYfgcb'g']bX]j ]Xi U''k cf\_"'7c''i g]cb'U'gc

# 7ci fgY GWX YXi `Y

## <u>J ]Yk "WUgg"hja YHUV"Y</u>

H]a YHJV`Y

8UHY	HmdY	7 cbhYbh
K_YY_"%"&" ">i `m'!"&- ">i `m	Hcd]W	
	@YWhifY	
	Hi hcf]U`	·
	Hcd]W	
5i [i gh	@YWn fY	
	Hi hcf]U`	
	Hcd]W	
5i [i gh	@YWn fY	
	Hi hcf]U`	
K YY_"(."% "5i[igh!"%	Hcd]W	
5i [igh	@YWn fY	
	Hi hcf]U`	
K YY_")."&\$"5i[igh!"&*	Hcd]W	
5i [i gh	@YWh fY	
	Hi hcf]U`	
K YY_'*.'&+'5i[igh!'&	Hcd]W	
GYdhYa VYf	@YWh fY	
	Hi hcf]U`	
	Dfc YWh	
K YY_'+.'' 'GYdhYa VYf'!'-	Hcd]W	
GYdhYa VYf	@YWh fY	)\$ Y
	Hi hcf]U`	
	-	
K YY_', .'%\$'GYdhYa VYf'! %*'GYdhYa VYf	A cXi `Y	%c
•	L	

	Hi hcf]U`	
K YY_'%&.'%) 'CWhcVYf'! &%'CWhcVYf	A cXi `Y	
&%°CWhcVYf	@YWn fY	
	Hi hcf]U`	

## FYgci fW/g

#### DfYgWf]VYX F Ygci fWYg

f&\$\$-Ł 5```ch\Yf`YggYbh]U``a UhYf]U`g`UfY hc`VY`Uj U]`UV`Y`cb`h\Y`Wci fgY`A ccX`Y"

#### FYWca a YbXYX FYgci fWyg

GYY'h\Y'Wti fgY'A ccX`Y"

## 7ci fgY 9j U`i Uh]cb UbX 8Yj Y`cda Ybh

H\Y`Wci fgYg`]g`dYf]cX]WJ``mfYj ]Yk YX`UbX`ghi XYbhg'`ZYYXVUW\_`]g`i gYX`hc`]a dfcj Y`]hg`WcbhYbhg`UbX UWijj ]hjYg'': YYXVUW\_`]g`[ Uh\YfYX`Zfca `ghi XYbhg`i g]b[ `a m9| dYf]YbWY''=n]g`YbWci fU[ YX`h\Uhghi XYbhg Wca d`YhY`h\Y]f`gi fj Ymg`VmUWWgg]b[ `h\Y`dYfgcbU`]gYX`k YV``]b\_`j ]U`h\Y`A ccX`Y`Wci fgY`g]hY''

#### =a U[ Y'7 fYX]h

| 'H]a Yg'BYk gdUdYfg'@]a ]hYX'&\$% 'O; cc[`Y'D\chc

#### 7F = 7CG

7F = CG'Dfcj]XYf'7cXY.'\$\$\$-,;