

MANF4100

8 Yg][b'UbX'5bU`mg]g'cZDfcXi WIDfcWYgg'GmgHYa g

HYfa 'C bY''##'&\$&%

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7 fYX]hDc]bhg *

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H\]g`Wci fgY j]bhY [fUhYg`h\Y`h\YcfmUbX`_bck `YX[Y`Zfca `Z]fghUbX`gYWcbX`mYUf`cZYb[]bYYf]b[`gW]YbWY Wci fgYg`]bhc`h\Y`h\YcfmUbX`dfUW]WY`cZXYg][b]b[`UbX`UbU`mg]b[`Ui hca UhYX`UbX`Wca di hYf!]bhY[fUhYX a Ubi ZJWi f]b[`gmghYa g"'H\Y`XYg][b`cZgi W.`gmghYa g`]g`Wca d`Yl `UbX`bYYXg`hc`Wcbg]XYf`a i `h]d`Y`ZJWhcfg UbX`XYg][b`Xf]j Yfgž`]bWi X]b[`h\Y`bUhi fY`UbX`W.UfUWNYf]gh]Wg`cZh\Y`dfcXi Whž'h\Y`a Uf_Yhž'h\Y`a Ubi ZJWi f]b[VfYUXh\`UbX`gWcdY`cZh\Y`cf[Ub]gUh]cbž'h\Y`Uddfcdf]UhY``Yj Y``cZa Ubi ZJWi f]b[`ZYl]V]`]mz`a Ubi ZJWi f]b[UbX`a UhYf]U`g`\UbX`]b[`hYW.bc`c[mUg`k Y```Ug`XYa c[fUd\]WW.UfUWNYf]gh]Wg`gi W.`Ug`fY[]cbU``k cf_ZcfWY g_]``g`UbX`\ci f`mdUmfUhYg`UbX`Ybj]fcba YbHJ``ZJWhcfg"`H\Y`dYfZcfa UbWY`cZgi W.`gmghYa g`bYYXg`hc`VY i bXYfghccX`]b`h\Y`YUf`mghU[Yg`cZWcbW ffYbhdfcXi WhdfcWYgg`XYg][b`UbX`Wcbh]bi ci g`mUXUdhYX`UbX]a dfcj YX`Ug`h\Y`bYYXg`UbX`fYei]fYa Ybhy'W.Ub[Y`h\fci [\ci hh\Y`dfcXi WhfUbX`dfcWYgg\ci`]ZY`WhWY"

7 ci fgY 5]a g

H\Y`Wci fgY`Vi]`Xg`cb`th\Y`_bck`YX[Y`cZth\Y`VUg]WY`Ya Ybhg`cZWca di hYf]nYX`a Ubi ZUWf fYž`UbX]b\WcfdcfUhYg`UbX`]bhY[fUhYg`th\]g`Z bXUa YbhU`_bck`YX[Y`]bhc`th\Y`XYg][b`cZWca di hYf]gYX`dfcXi Wf]cb ghfUhY[]Yg`UbX`gmghYa g"`H\]g`Wci fgY`k]```ZcWi g`cb`th\Y`hcc`gž`UbU`mgYg`UbX`ZfUa Yk cf_g`Zcf`gY`YWf]b[`UbX XYg][b]b[`Wca di hYf]gYX`UbX`Ui hca UhYX`dfcXi Wf]cb`gmghYa gž`]bWi X]b[`th\Y`Z bXUa YbhU`g`cZZYI]V`Y

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8 YHJ]`g. H\Y`Ugg][ba Ybh]bghi Wjcbg`k]```VY`dcghYX`cb`A ccX`Y#HYUa g`cf`\UbXYX`ci h]b`WUggž`UbX`U fYa]bXYf`Ubbci bWa Ybhk]```VY`a UXY`UVci hXi Y`XUhY`Zcf`h\Y`Ugg][ba Ybhg"'H\Y`Ugg][ba Ybhgʻgi ddcfhi\Y`YUfb]b['ci hWa Yg`Vm]bWafdcfUhjb['Ub`Uddfcdf]UhY`a]l 'cZUWlj]hjYgʻgi W\`Ug`]ggi Y`UbU`mg]g`UbX`ZUWl VUgYX`XUhU`UbU`mg]g`h\Uhgi ddcfhi\Y`XYg][b`cZUddfcdf]UhY`gc`i hjcbg`UbX`ghfUhY[]Yg"'H\Y`Ugg][ba Ybhg U`gc`gi ddcfhi\X``UVcfUhjj Y`hYUa k cf_`UbX`]bhY[fUhjcb`cZX]ZYfYbh]XYUg`UbX`Wa dcbYbhg`]bhc`Ub`cj YfU`` Wc\YfYbhei U`]hma UbU[Ya YbhghfUhY[m"

 $\label{thm:limit} \begin{array}{ll} \mbox{Hi fb]h]b"gYhh]b" . "H\]g"Ugg][ba Ybh"]g"gi Va]hhYX"h\]fci [\]'Hi fb]h]b"UbX"gh XYbhg"Xc"bchigYY"Hi fb]h]b g]a]"Uf]hmfYdcfhg" \\ \end{array}$

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8 YHU]`q. H\Y`ei]n`k]``VY`WcbXi WYX`Z ``mcb`]bY`i q]b['A ccX`Y`Ei]n`Z bWf]cb"

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		G]a d'Yl 'A Yh\cX; Yca YhfmcZh\Y'G]a d'Yl 'A Yh\cX
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K YY_', .') '5df]`'!'- '5df]`	@YVVi fY	5[[fY[U hǐz f

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 $C \, dYf Uhlcbg `A \, UbU[\, Ya \, Ybh'- `Gi \, ghU]bUV]`]hmUbX `Gi \, dd`m7 \, \ \ U]b `A \, UbU[\, Ya \, Ybh'z > "`< Y]nYf`UbX `6 "`F YbXYfz & $\% ZDYUfgcb `9Xi \ Whlcb "`- +, \&- &\((, *) * \)$

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±b h\]g Wci fgYžfYWbh]a dfcj Ya Ybhg fYgi `h]b['Zfca 'ghi XYbhZYXVUW_']bWi XY h\Y Wcbh]bi ci g i gY cZJ ≠ 5 UggYgga YbhUbX'ZYYXVUW_'Ug']hfYWY]j YX'dcg]h]j Y'ZYYXVUW_k\Yb h\Y Wci fgY k Ug'fYj]gYX'g]bW' &\$% "

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UggYgga YbhidYfZcfa UbWYžmci 'UfY'Y`][]V`Y`hc'Udd`m'Zcf'GdYWJU``7cbg]XYfUhJcb'df]cf'hc'gi Va]htjb['UbUggYgga Ybhicf'g]htjb['Ub'Yl Ua"

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D`YUgY'bchY'h\Uhghi XYbhg'k]``bch'VY'fYei]fYX'hc'dfcj]XY'Ubm'XcW'a YbhUfm'Yj]XYbWY'hc gi ddcfh'UVgYbWg'Zfca 'Ubm'WUggYg'a]ggYX'**VYWU gY'cZ'7CJ=8!%** 'di V`]W\YU'h\ 'a YUgi fYg'gi W\'Ug]gc`Uh]cb"'| BGK 'k]``bch'VY']bg]gh]b['cb'a YX]WD`'Wfh]ZJWUh'g'Zfca 'UbmcbY'XYYa YX'hc'VY'U'dcg]h]j Y'WUgYž cf'k\Yb'h\Ym\Uj Y'fYWcj YfYX"'Gi W\'Wfh]ZJWUh'g'UfY'X]ZZJW 'hhc'cVhU]b'UbX'di h'Ub'i bbYWfggUfmghfU]b'cb ghi XYbhg'UbX'a YX]WU`'ghUZZ''

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5 WUXYa]W<cbYghmUbX D`U[]Uf]ga

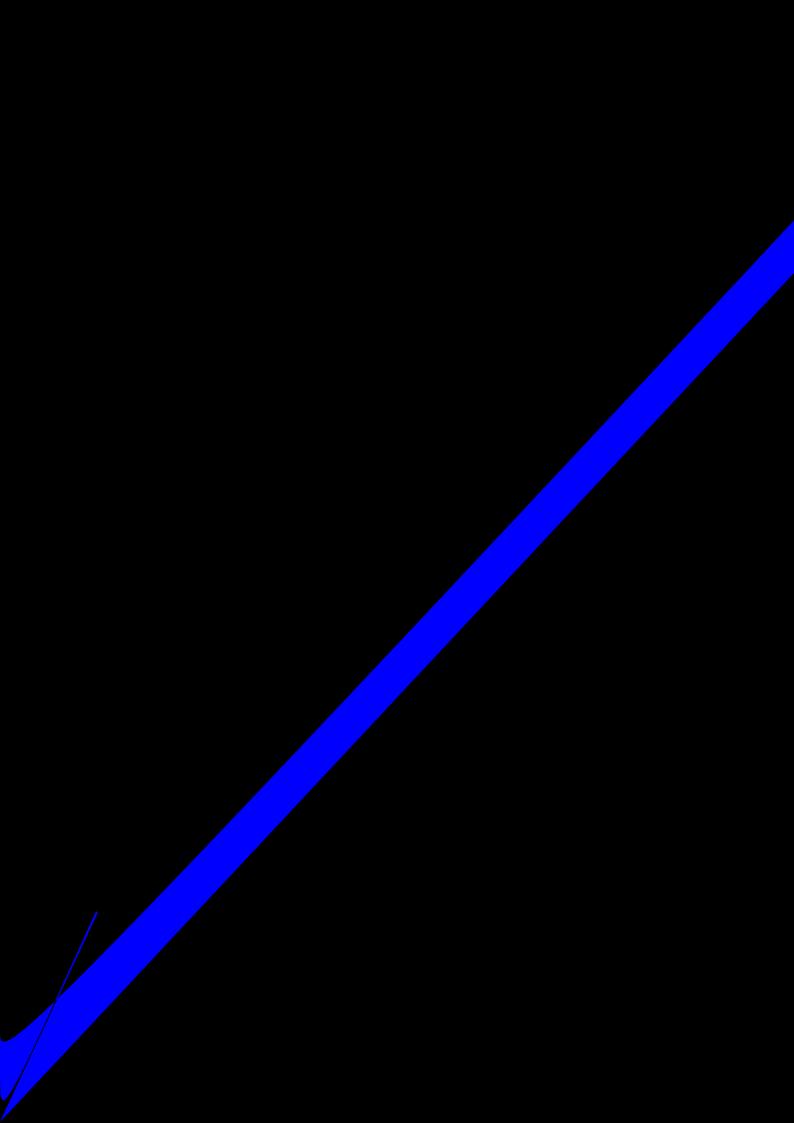
 $\label{thm:continuous} IBGK ``Ug`Ub`cb[c]b[`Wcaa]haYbhhc`ZcghYf]b[``U`W``hifY`cZ`YUfb]b[`]bZcfaYX`VmUWUXYa]W]bHY[f]hm'`5``IBGK `ghiXYbhg`\UjY`U`fYgdcbg]V]`]hmhc`UX\YfY`hc`h\]g`df]bW]d`Y`cZUWUXYa]W]bHY[f]hm'D`U[]Uf]gaibXYfa]bYg`UWUXYa]W]bHY[f]hmUbX`]g`bchhc`YfUhYX`UhIBGK "`B'SY[Sde_SfG@EI[eVW{`WSege[YfZW_aXSekagdai`ž$

Mci 'UfY'U'gc'fYa]bXYX'h\UhWUfYZ ``h]a Y'a UbU[Ya Ybh]g'Ub']a dcfhUbhdUfhcZgh XmUbX'cbY'cZh\Y]XYbh]Z]YX'WUi gYg'cZd`U[]Uf]ga ']g'dccf'h]a Y'a UbU[Ya Ybh'Gh XYbhg'g\ci `X'U``ck 'gi Z]W]Ybh'h]a Y'Zcf fYgYUfW\žXfUZ\]b['UbX'h\Y'dfcdYf'fYZYfYbW]b['cZgci fW/g']b'dfYdUf]b['U```UggYgga Ybh'hUg_g''

₹d`U[]Uf]ga i]gʻZci bXi]bimci fik cf_ik \Ybimci i'UfYi]b iZ[fghmYUfžimci fiYVWi fYfik]``cZYfimci i'Ugg]gHUbWYihc
]a dfcj Yimci fiUWUXYa]Wg_]``g'''H\Yma UmUg_imci ihci`cc_'Uhigca Yicb`]bYifYgci fWygžiUhhYbXih\Yi@YUfb]b[
7 YbhfYžicfigca Yh]a YgifYgi Va]himci fik cf_ik]h\h\YidfcV`Ya iZ[l YX'''< ck Yj Yfia cfYigYf]ci gi]bghUbWygi]biZ[fgh
mYUfžigi W\ugighYU`]b[iUbch\Yfigh XYbhgik cf_icfidUm]b[igca YcbYihciXcimci fik cf_žia UmVYi]bj Ygh][UhYX
i bXYfih\YiGh XYbhA]gWcbXi WhDfcWYXi fYg''

FYdYUHYX'd`U[]Uf]ga 'fYj Yb']b'Z|fghmYUfłź'd`U[]Uf]ga 'UZHYf'Z|fghmYUfźcf'gYf]ci g']bghUbWYgža Um'U`gc'VY]bj Ygh][UhYX'i bXYf'h\Y'Gh XYbhA]gWcbXi WhDfcWYXi fYg"'H\Y'dYbU'h]Yg'i bXYf'h\Y'dfcWYXi fYg'WUb']bWi XY U'fYXi Whcb']b'a Uf_gž'ZU]`]b['U'Wci fgY'cf'Zcf'h\Y'a cghgYf]ci g'a UhHYfg'fi]_Y'd`U[]Uf]ga ']b'Ub'\cbci fg h\Yg]gŁYj Yb'gi gdYbg]cb'Zfca 'h\Y'i b]j Yfg]mm''H\Y'Gh XYbhA]gWcbXi WhDfcWYXi fYg'UfY'Uj U]`UV'Y'\YfY.

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DfcZYgg]cbU`'UbX'dYfgcbU`'Ultf]Vi hYg	
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