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MATHEMATICS ENRICHM[redacted]T CLUB.
Problem Sheet 13 Solutions, Se[redacted]mber 3, 2019

1. We can write

$$y = 1000 \quad x \text{ [redacted]}$$
$$= 1001x \text{ [redacted]}$$

Also, $y = kx^2$ for some integer k. Thus

$$1001x = kx \text{ [redacted]}$$

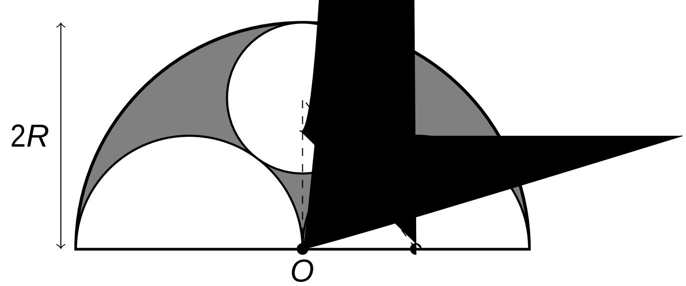
Since $x \neq 0$, we may divide both sides by x to o[redacted]

$$kx = 1001 \text{ [redacted]}$$
$$= 7 \quad 14 \text{ [redacted]}$$

Since x is a three-digit number, $x = 143$ and y [redacted]143.

2. If we calculate the fth powers in mod 10, w[redacted]that 1^5 1 (mod 10), 2^5 2 (mod 10), 3^5 3 (mod 10), and so on. That is[redacted]last digit of x^5 will be the same as the last digit of x. Therefore, the last digit o[redacted] $2^5 + \dots + 2019^5$ is equal to the last digit of $1 + 2 + \dots 2019$, which is 0.

3. Let O be the centre of the big semi-circle; A be[redacted]entre of one of the smaller semi-circles, and B be the centre of the inscribed cir[redacted]



Clearly, $OA = R$. Now AB is a straight line through the point of tangency of the circle and the semi-circle, so $AB = R + r$. Since the two smaller semi-circles are congruent, and the circle is tangent to both of them, B lies on the vertical axis of symmetry of the larger semi-circle. Thus $OB = 2R - r$.

Thus, by Pythagoras' theorem,

$$\begin{aligned} (R + r)^2 &= (2R - r)^2 + R^2 \\ R^2 + 2rR + r^2 &= 4R^2 - 4rR + R^2 + R^2 \\ 6rR &= 4R^2 \\ 3r &= 2R \end{aligned}$$

Hence $R : r = 3 : 2$, as required.

4. Suppose we can place the numbers on a circle so that the condition holds. Let us call the integers from 26 to 75 *normal*, and all the others *extreme*. Two extreme integers cannot be consecutive (their difference is either less than 25 or greater than 50). Note that the numbers of the extreme and normal integers are the same and therefore they must alternate. However the normal number 26 can be adjacent to only one extreme integer 76, which is a contradiction.

5. Let $a = 10^b$, then we can rewrite the inequality $10 < a^x < 100$ as $1 < bx < 2$. Similarly, if $100 < a^x < 1000$, then $2 < bx < 3$. Suppose n is the smallest integral solution to the inequality, then since there are exactly 5 solutions, the largest solution must be $n + 4$. From this we can deduce $b(n - 1) < 1 < bn$ and $b(n + 4)$

Now, consider $4ABH$. As AH is a diameter of the circle, $\backslash HBA =$

$\{k_1, k_2, \ldots, k_a\}$ and $K_2 \{k_{a+1}, k_{a+2}, \ldots, k_n\}$, then we can change the anti-clockwise ordering of the seated knights into an clockwise ordering, by reversing the order of the knights in the set K_1 and K_2. To move k_1 to s_a, k_1 must swap position with k_2 then k_3 and so on. It takes $(a - 1)$ swaps to move k_1 to the seat s_a. Similarly, it takes $(a - 2)$ swaps to move k_2 into s_{a-1}, $(a - 3)$ swaps to move k_3 to s_{a-2} and so on. Therefore, it takes $1 + 2 + \ldots + (a - 1)$ swaps to reverse the order of the set K_1. Similarly, it takes $1 + 2 + \ldots + (n - a$