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MATHEMATICS ENRICHMENT CLUB.

Solution Sheet 4, June 3, 2019[1]

1. (a) Note that $8 = 2^3$ has a remainder of one when divided by 7. That is, we say that $2^3 \equiv 1 \bmod 7$. In modular arithmetic, if $a \equiv b \bmod n$ then $a^x \equiv b^x \bmod n$. Since $2^{2019} = 2^{3 \times 673} = 8^{673}$, we can conclude that 2^{2019} has remainder 1 when it is divided by 7.

   (b) Let's look at the last digit of 2^n. That is, we will find $2^n \bmod 10$ for $n = 1, 2, 3, \ldots$.

$$
\begin{aligned}
2^1 &\equiv 2 \bmod 10\\
2^2 &\equiv 4 \bmod 10\\
2^3 &\equiv 8 \bmod 10\\
2^4 &\equiv 6 \bmod 10\\
2^5 &\equiv 2 \bmod 10\\
2^6 &\equiv 4 \bmod 10\\
&\vdots
\end{aligned}
$$

   We can see that this four-step pattern will keep repeating. Thus we can write

$$2^{2019} = 2^{4 \times 504 + 3} \equiv 8 \bmod 10.$$

   Thus the last digit is 8.

2. Gerald can roll either $\{1, 2, 3, 4, 5\}$ or four-step pattern will keep repeating. Thus we can write

2^{2019} 2019

3.

$$\sqrt{x + \sqrt{y + \sqrt{x + \sqrt{y + \cdots}}}} = 7$$

$$\sqrt{y + \sqrt{x + \sqrt{y + \cdots}}} = 7^2 - x$$

$$\sqrt{x + \sqrt{y + \cdots}} = (49 - x)^2 - 2$$

$$7 = (49 - x)^2 - 2$$

$$x = 46.$$

4.

b from being a divided by 3, the numbers 38 and 44 must be in a different set to 46, thus we have two possibilities so far

$$A = \{38, 39, 44\}, B = \{45, 46\} \qquad \text{or} \qquad A = \{39, 46\}, B = \{38, 44, 45\}.$$

For the first case, the sum of A is 121 and the sum of B is 91, so we must add 24 to one set and 48 to the other. However, adding 24 or 48 to A does not give a prime number.

For the second case, the sum of A is 85 and B is 127. Since 85 is not prime, we must add 24, 48 or both to it. If we add 24 to 85, then we have to add the other number 48 to 127, which gives 175. But 175 is not a prime so this is not a solution. If we add 48 to 85, we get 133 which is not prime. Therefore the only solution is $\{24, 39, 46, 48\}$, $\{38, 44, 45\}$.

6. The last digit of x must be 0, because we need an integer after increasing x by 10%. Now in order to decrease the sum of digits of a number after increasing it by 10%, we look for a number that has a lot of digit that will "carry" up when multiplied by 1.1; an example would be a number starting with m lots of 3's, with n lots of 6's in the middle and a 0 at the end, because increasing this number by 10% gives a number with the $n-1$ lots of 3's, $m-1$ lots of 6's, one of each 7, 2 and 0. So the equation we need to solve is $(n-1)3 + (m-1)6 + 9 = 0.9(3m + 6n)$; $n = m = 10$ is a solution.

Senior Questions

1. The trick is to apply a change of variable, so that the two graphs become symmetrical.