Parabola Volume 57, Issue 3 (2021)

59th Annual UNSW School Mathematics Competition: Competition Problems and Solutions

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A Junior Division – Problems

Problem A1:
Three grasshoppers play the following leapfrog game. They start off at three vertices of a square. At each step, a grasshopper leaps over another one and lands at the point

B Senior Division – Problems

Problem B1:
An ant crawls on a table with constant speed in one direction, then every 15 minutes changes direction by turning 90^0. Prove that the ant can only return to the original position after a whole number of hours have elapsed.

Problem B2:
The Fibonacci sequence is $1; 1; 2; 3; 5; 8; 13; \ldots$; or, formally, the sequence $F = \{f_n\}_{n=1}^{\infty}$ of integers which is defined by the following recurring relation

$$f_1 = 1; \quad f_2 = 1 \quad \text{and} \quad f_n = f_{n-1} + f_{n-2}; \quad n \geq 3.$$

Find the remainder when f_{2021} is divided by 19.

Problem B3:
An island inhabitants design an emergency response service based on helicopters. Their island is a disc of radius 100km and they plan to purchase helicopters capable of flying 300km/h. Find the minimum number helicopters needed for the emergency service in order to be able to reach every point of the island within 10min. Assume that helicopter's takeoff and landing times are negligible.

Problem B4:
Prove that the number $4n^4 + m^4$ is not prime if $m; n$ are positive integers and $m \neq n$.

Problem B5:
A company has one Director, ten Senior Managers, one hundred Site Supervisors, and

A Junior Division – Solutions

Solution A1.

Answer: No.

Assume that a coordinate system is given such that the original positions are $(0; 0)$, $(0; 1)$ and $(1; 0)$. If two grasshoppers have the coordinates $(x; y)$ and $(a; b)$ and the one at $(x; y)$ takes the leap, then it lands at $(2a - x; 2b - y)$.

We infer that (a) the grasshoppers will always stay on integer coordinates; and that (b) if $x + y$ is odd at starting position, then $x + y$ is odd at the landing position.

Hence, no grasshopper will land at even position $(1; 1)$. □

Solution A2.

Answer: 63.

Let the number of players be n and let a_1 the number of pokemons caught by the 1st player, a_2 the number of pokemons caught by the 2nd player and so on. Assume that

$$a_1 < a_2 < \cdots < a_n.$$

It is known that $a_k \geq 1$, for every $k = 1, 2, \ldots, n$. Hence, $a_{k+1} \geq a_k + 1$ for $k = 1, 2, \ldots, n-1$; or $a_1 \geq 1$, $a_2 \geq 2$, ..., $a_n \geq n$.

Therefore,

$$2021 = a_1 + a_2 + \cdots + a_n \geq 1 + 2 + \cdots + n = \frac{(n+1)n}{2}$$

or, in other words, $n^2 + n \leq 4042$. Solving this inequality gives $n \leq 63$. □

Solution A3.

The player who picks second wins. One possible winning strategy is the following.

After the first player makes his pick; the second player ensures that, after his pick, the remaining petals are arranged in two groups of adjacent petals with the same number of petals.

Subsequently, after each move of the second player, he ensures that the petals are arranged in even number of groups, and each group has a matching pair with the same number of petals. □

Solution A4.

The number 133 factors 7×19.

The first eight mod 7 remainders of the F sequence are

$$1, 1, 2, 3, 5, 1, 6, 0.$$

Note that $f_8 \equiv 0 \mod 7$. Let's prove that $f_{8k} \equiv 0 \mod 7$ for every $k \in \mathbb{N}$. The next two elements in the sequence mod remainders mod 7 are 6 and 6. Hence, the eight remainders, from 9-th to 16-th are

$$6 \times (1, 1, 2, 3, 5, 1, 6, 0) \mod 7 = 6, 6, 5, 4, 2, 6, 1, 0.$$

Hence, the 17th and 18th elements are 1 and 1 and the sequence of remainders repeat from this point on.

The first eighteen elements of mod 19 remainders are

$$1, 1, 2, 3, 5, 8, 13, 2, 15, 17, 13, 11, 5, 16, 2, 18, 1, 0.$$

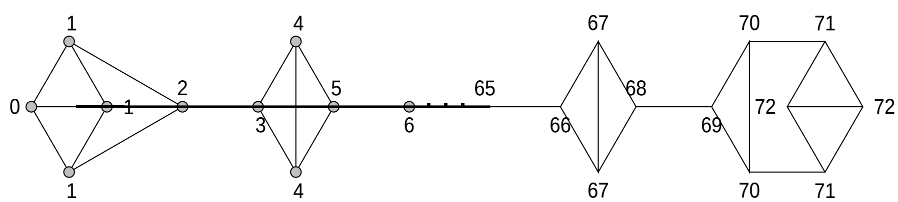
The next two elements are 1 and 1 and the sequence of remainders repeats.

Hence, $f_{18k} \equiv 0 \pmod{19}$ for every $k \in \mathbb{N}$. Since $\mathrm{lcm}(8, 18) = 72$, we have $f_{72k} \equiv 0 \pmod{19 \times 7}$ for every $k \in \mathbb{N}$. □

Solution A5.

Answer: Yes.

The acquaintances graph is shown below. Each dot represents a person and each edge represents acquaintance relation. Each integer is the day after 1 Jan ('1' is 1 Jan) the person gets infected. The integer which corresponds to 5 Mar is '64' and the integer which corresponds to 14 Mar is '73'.



Let k be the total number of horizontal components and m be the total number of vertical components. That is,

$$\vec{H} = \vec{h}_1 + \dots + \vec{h}_k \quad \text{and} \quad \vec{V} = \vec{v}_1 + \dots + \vec{v}_m .$$

Since horizontal and vertical components alternate, we have $k = m$ or $k = m \pm 1$. On the other hand, the ant crawls with constant velocity, so both numbers m and k are even. Therefore, $m = k = 2s$ for some $s \in \mathbb{N}$ or $m + k = 4s$.

Hence, the total travel time is $15 \times (m + k) = 60 \times (m + k)$; or $(m + k)$-hours. □

Solution B2.

The first eighteen elements of mod 19 remainders are

$$1, 1, 2, 3, 5, 8, 13, 2, 15, 17, 13, 11, 5, 16, 2, 18, 1, 0 .$$

The next two elements are 1

Solution B4.

$$\begin{aligned} 4n^4 + m^4 &= (\sqrt{2}n)^4 + m^4 \\ &= (2 \end{aligned}$$