MATHEMATICS ENRICHMENT CLUB.[1]
Problem Sheet 12, August 13, 2013

1. Solve $\dfrac{x+3y}{2x+5y} = \dfrac{4}{7}$.

2. If a, b, c, d, e are real numbers such that

$$a - b + c = 1;\quad b - c + d = 2;\quad c - d + e = 3;\quad d - e + a = 4 \text{ and } e - a + b = 5;$$

which of a, b, c, d, e is the largest?

3. Is it possible to cut a square into nine squares and colour one of them white, three of them grey and five of them black, such that squares of the same colour have the same size and squares of different colours will have different sizes?

4. Take any triangle ABC and show how to construct an equilateral triangle inside ABC whose vertices touch the sides of ABC. (Hint: Start by constructing an equilateral triangle outside ABC with AB as one of its sides.)

5. 100 Queens are placed on a 100×100 chessboard so that no two attack each other (Queens attack each other if they are both on the same row, coloumn or diagonal). Prove that each of the four 50×50 corners of the board contains at least one Queen.

6. (a) Verify that

$$\begin{aligned} x^{15} - 1 &= (x^3 - 1)(x^{12} + x^9 + x^6 + x^3 + 1) \\ &= (x^5 - 1)(x^{10} + x^5 + 1). \end{aligned}$$

   (b) Hence factor $2^{15} - 1$ as a product of prime factors.

   (c) Can you factorise $2^{15} + 1$ as a product of prime factors?

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[1] Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni. Senior problem 1 provided by I. Woodhouse. Some problems are from the Tournament of Towns

Senior Questions

1. Let $z = \cos\theta + i\sin\theta$, $z \neq 0$.

   (a) Show that for

   $$z^n + \frac{1}{z^n} = 2\cos n\theta\,;$$

   for $n = 0; 1; 2; \ldots$.

   (b) Show that

   $$\left(z + \frac{1}{z}\right)\left(z^{n-1} + \frac{1}{z^{n-1}}\right) - \left(z^{n-2} + \frac{1}{z^{n-2}}\right) = z^n + \frac{1}{z^n};$$

   for $n = 1; 2; 3; \ldots$.

   (c) Hence deduce that

   $$\cos((n-2)\theta) = 2\cos\theta\cos((n-1)\theta) - \cos n\theta\,.$$

2. How many real roots does the equation $x = 3\pi(1 - \sin x)$ have? Use Newton's method to find an approximate value of the smallest one and hence find the largest one.