

MATHEMATICS ENRICHMENT CLUB.¹
Problem Sheet 3, May 21, 2013

1. The perimeter of a base of a rectangular brick with integer sides is 18 cm, whilst its volume is 42 cm³. What is its height?

2. Calculate

$$1 - \frac{1}{2} + 1 - \frac{1}{3} + 1 - \frac{1}{4} + \dots + 1 - \frac{1}{2008} = ?$$

3. Find the smallest positive integer whose square ends in (a) 09 and (b) 9009.

4. Show that if a, b are positive numbers such that $ab = 1$ then

$$\frac{a}{b+1} + \frac{b}{a+1} + (1-a)(1-b) = 1;$$

5. Suppose we have the numbers $x_0 = 0; x_1 = 1$ and $x_{n+1} = x_n + 2x_{n-1}$ for $n \geq 2$.

(a) Write down the numbers x_n for $n = 2; 3; 4; 5; 6$.

(b) Show that there is no n for which $x_n = 1999$. (Hint: Use modulo 8 arithmetic).

(c) Show that $x_n = \frac{2^n - (-1)^n}{3}$ satisfies the equation.

6. In $\triangle ABC$, extend the sides AB and AC and draw a circle outside the triangle which touches BC and these two produced sides. This circle is called the *escribed circle* of the triangle.

(a) Show that $r_1 = \frac{A}{s-a}$, where r_1 is the radius of the escribed circle, A is the area of $\triangle ABC$, a is the length of BC and s is half the perimeter of $\triangle ABC$.

(b) Show that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$ is the radius of the incircle (recall last week's result.)

7. $ABCD$ is a parallelogram, O a point inside it. Prove that the sum of the areas of $\triangle AOB$ and $\triangle COD$ is half the area of $ABCD$.

¹Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.

Senior Questions

1. Prove that the square of the n th triangle number is the sum of the first n cubes, i.e.

$$\sum_{k=1}^n k^2 = \sum_{k=1}^n k^3; \quad \text{for } n \geq 1.$$

2. Find the limit $\lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n^3}$.

3. A hand of eight cards is dealt from a standard pack. How many hands contain exactly three cards of the same value and the remaining cards from the remaining suit?