

MATHEMATICS ENRICHMENT CLUB.¹
Solution Sheet 4, May 28, 2013

- (a) Writing $(21)_b$ and $(12)_b$ in base ten then we must have $2b+1 = 2(b+2) \Rightarrow 1 = 4$, a contradiction.

(b) In base ten we must satisfy $ab + c = 2(cb +$

Writing $b = 3m + 2$ we can replace all bs with ms and get

$$a = (2m + 1)k$$

$$c = mk$$

$$k < \frac{3m + 2}{2m + 1}; \quad k \geq \mathbb{N}; \quad m \geq \mathbb{N};$$

2. Write $1000 = \sum_{k=a}^b (2k - 1)$, $0 < a < b$ which is an arithmetic progression, so reduces to $1000 = (b - (a - 1))(b + (a - 1))$. So now we look for two numbers $x = b; y = a - 1$ whose sum and difference are both factors of 1000. The factors of 1000 are (1;1000), (2;500), (4;250), (8;125), (10;100), (20;50), (25;40). Since we must have one factor represented by $x - y$ and the other by $x + y$, both factors must be even, which leaves 4 possible pairs for x and y , and hence 4 pairs b and a (since $a = b$).

Finally, if 1000 is the sum of consecutive, positive odd numbers $k + (k + 2) + (k + 4) + \dots$, we can also add all the odd integers $(k - 2); (k - 4); \dots; 3; 1; 1; 3; \dots; (k - 2)$ without changing the original value. So for each of the 4 pairs a and b above, we have another representation. Hence, I count 8 ways.

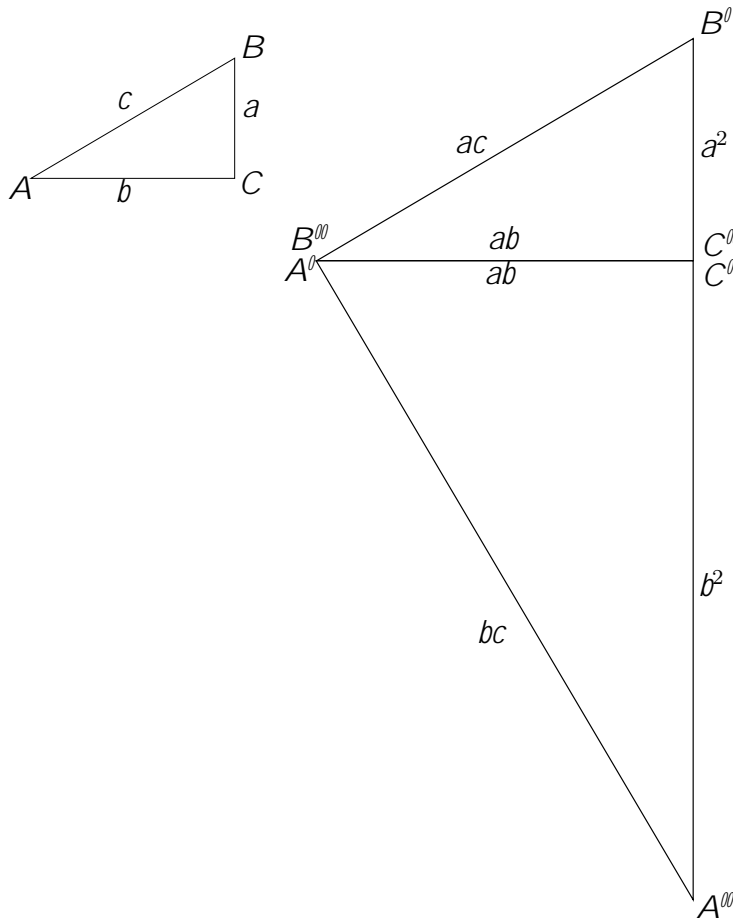


Figure 1: Picture for question 3

3. The new triangle is $A''B''C''$, where C''

triangle $A''''B''''C''''$ is the enlargement of ABC by a factor of c , which implies $a^2 + b^2 = c^2$; Pythagorus' Theorem.

4. Angle A is 10° and angle C is 30° .
5. (a) The triangles BAD and KAL are similar since they have two sides in ratio ($AK : AB = 1 : 3$ and $AL : AD = 1 : 3$) which contain the common angle A . For the same reasons, triangles BCD and NCM are similar. Thus KL is parallel to BD which is parallel to MN . Also, the lengths $BD = 3KL$ and $BD = 3MN$ so $KL = MN$. Thus $KLMN$ is a parallelogram because it has one pair of equal length and parallel sides.
- (b) In the same fashion as above we show that the triangles ABC and KBN are similar, and that the triangles ADC and LDM are similar, with $AB : KB = 3 : 2$, $BC : BN = 3 : 2$, $AD : LD = 3 : 2$ and $DC : DM = 3 : 2$. Thus the areas are in the ratios $\text{area}(ABD) : \text{area}(AKL) = 1 : 9$, $\text{area}(BCD) : \text{area}(NCM) = 1 : 9$, $\text{area}(ABC) : \text{area}(KBN) = 4 : 9$ and $\text{area}(ADC) : \text{area}(LDM) = 4 : 9$. From this we obtain the two equations

$$\text{area}(ABCD) = \text{area}(ABD) + \text{area}(BCD) = 9 \text{area}(AKL) + 9 \text{area}(NCM)$$

$$\text{area}(ABCD) = \text{area}(ABC) + \text{area}(ADC) = \frac{9}{4} \text{area}(KBN) + \frac{9}{4} \text{area}(LDM);$$

combining which yields

$$\text{area}(ABCD) + 4 \text{area}(ABCD) = 9(\text{area}(AKL) + \text{area}(NCM) + \text{area}(KBN) + \text{area}(LDM))$$

$$5 \text{area}(ABCD) = 9(\text{area}(ABCD) + \text{area}(KLMN))$$

$$9 \text{area}(KLMN) = 4 \text{area}(ABCD)$$

$$\text{area}(KLMN) = \frac{4}{9} \text{area}(ABCD):$$

6. Consider a right triangle with perpendicular sides of length m and n , and thus hypotenuse $\sqrt{m^2 + n^2}$. Say one of the non-right angles is θ then

$$\frac{m+n}{\sqrt{m^2+n^2}} = \cos \theta + \sin \theta :$$

The value $\cos \theta + \sin \theta$ is maximum when $\cos \theta = \sin \theta$, and so when $\cos \theta = \sin \theta = \frac{1}{\sqrt{2}}$. Thus

$$\frac{m+n}{\sqrt{m^2+n^2}} = \frac{2}{\sqrt{2}} = \sqrt{2}:$$

Senior Questions

1. The radii of the inscribed circle that touch the triangle divide the triangle into 2 pairs of congruent triangles and a square. Thus the perimeter of the triangle is $p = 2x + 2y + 2r$ where $x + y = 15 \text{ cm}$ is the length of the hy-

