MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 7, June 17, Brazil 2014 [1]

1. Let's simply start by substituting x into the equation given, so

$$x^3 \quad 3x$$

2. In the case where the deleted vertex is part of a triangle, two vertices of the triangle merge to become one, so $v \to v - 1$, two edges disappear, the one connecting the two merged vertices disappears while the other two merge to become one, so $e \to e - 2$ and finally, the face formed by the triangle disappears so $f \to f - 1$. If the deleted vertex is not part of a triangle, its either part of a lone line, or a higher order polygon. If a lone line, then we simply see one vertex disappear and the edge that connected the two along with it, so $f \to f$, $v \to v - 1$ and $e \to e - 1$. If part of a higher order polygon, again the number of faces stays the same and we just lose one edge and one vertex, so again $f \to f$, $v \to v - 1$ and $e \to e - 1$.

As we can see from the two cases above, $v - e + f$ is constant when we delete a vertex by merging it with a neigbour (in the first case $v - e + f \to v - 1 - (e - 2) + f - 1 = v - e + f$ and in the second $v - e + f \to v - 1 - (e - 1) + f = v - e + f$). So we can delete all but one vertex in this manner, all while $v - e + f$ is constant. A graph on a sphere with only one vertex has $v = 1$, $e = 0$ and $f = 1$ so $v - e + f = 2$.

Suppose we have p pentagons and w hexagons. yall , Australia cannot advance with 2, 1 or 0 points.

The most intensive situation for us is if Austrlia gets 4 points, in which the short answer is maybe.

We can start by writing down an upper bound for the probability:

$$\mathbb{P}(\text{Australia advances}) < \mathbb{P}(\text{Aust gets 6 points}) + \mathbb{P}(\text{Aust gets 4 points}) + \mathbb{P}(\text{Aust gets 3 points})$$
$$= \frac{1}{13}\frac{1}{15} + \frac{1}{13}\frac{1}{10} + \frac{8}{65}\frac{1}{15} + \frac{1}{13}\frac{5}{6} + \frac{4}{5}\frac{1}{15}$$
$$\mathbb{P}(\text{Australia advances}) < \frac{9}{65}:$$

We could then refine this by including the appropriate conditional probabilities, for instance the probability that Australia advances by "beating the Netherlands, have Chile win all 3 games, then win the coin toss against the Netherlands" is

$$\mathbb{P} = \mathbb{P}(A \text{ beats } N)\mathbb{P}(S \text{ beats } A)\mathbb{P}(C \text{ beats } S)\mathbb{P}(C \text{ beats } N)\frac{1}{2}$$

which makes up one part of $\mathbb{P}(\text{Australia advances with 3 points}) < \mathbb{P}(\text{Australia gets 3 points})$.

4. This question was taken from UNSW's *Parabola* but the solution is missing a diagram, the website `http://mathafou.free.fr/pbg_en/sol110b.html` does an excellent job of showing the same construction with an accompanying diagram.

5. The two triangles $\triangle KXA$ and $\triangle DXC$ are similar as the three corresponding angles are equal. Suppose that $AK = \ell$ then since K is the midpoint of AB, $DC = 2\ell$. Let the height of rectangle, BC, be h, and the area of $\triangle KXA = a$. Since $\triangle KXA$ and $\triangle DXC$ are similar with side ratios 1 : 2 then they have areas in ratio 1 : 4. If the perpendicular height of $\triangle KXA$ is y, then the sum of their areas can also be written as $\frac{1}{2}\ell y + \frac{1}{2}(2\ell)(h - y)$ so

$$\begin{aligned} a + 4a &= \frac{1}{2}\ell y + \frac{1}{2}(2\ell)(h - y) \\ &= \frac{1}{2}\ell y + \ell h - \ell y \\ &= \ell h - \frac{1}{2}\ell y = \ell h - a \\ 6a &= \ell h \\ 12a &= (2\ell h) \end{aligned}$$

where $2\ell h$ is precisely the area of the rectangle. So $\triangle KXA$ takes up $1/12$th the area of $ABCD$.

Senior Questions

1. This is known as the *intermediate value theorem* and is essentially the result of the real numbers 'having no gaps', and that continuous functions can't 'jump'. What we

mean by continuous functions not being able to `jump' is that if x_1 and x_2 are close together, then so are $f(x_1)$ and $f(x_2)$. In symbols, we say if $|x_1 \quad x_2$

4. This was supposed to read \Let g be as above except $g(a) \neq g($