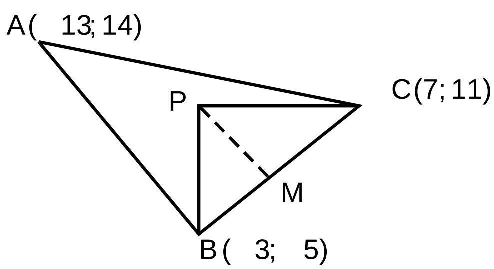
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 12, August 2, 2015

1. (a) The remainder of 2017 divided by 3 is 1, so the remainder of 2017^{46} divided by 3 is $1^{46} = 1$. Also, the remainder of 46 when divided by 3 is 1. Hence, the remainder of 2017^{46} 46 divided by 3 is zero; that is 2017^{46} 46 is not prime.

   (b) Apply the same technique as in part (a), we can show that $2017^{46} + 46$ is divisible by 5.

2. Let $f(x) = x^3 + px^2 \quad x + q$. Then since $(x \quad 5)$ is a factor of $f(x)$, $x = 5$ is a root of $f(x)$ which means $f(5) = 125 + 25p \quad 5 + q = 0$. Furthermore, $f(x)$ has remainder 24 when divided by $(x \quad 1)$ which means $f(1) = 1 + p \quad 1 + q = 24$. We now have two equations of p and q, thus we can solve simultaneously to obtain $p = 6$ and $q = 18$.



3. Let M be the point of interception between the line BC and the line perpendicular to BC and passing through the point P; see above. Then since 4

It is clear from the diagram that $x = -8\sqrt{3}$ else p would be outside of $\triangle ABC$, hence $y = 5\sqrt{3}$. Therefore the coordinate of the point P is $(2 - 8\sqrt{3}, 2 + 8\sqrt{3})$.

4. The answer is 3; The sequence is $\ldots 3, 4, 5, 6, 7, 8, 9, \ldots$, the sum of the middle five terms are equal to the sum of the last four. So to complete this problem, we just need to show that there is no way to get less than 3 overlaps.

   Firstly, the sum of five consecutive terms is 30, the sequence must contain 6. Also, the sequence cannot be constant, because if it is then the sum of four consecutive terms is 24. Therefore, the common difference $d \geq 1$.

   If the overlap is less than three terms, then the sum of the four consecutive terms cannot include 6. Additionally, the four consecutive terms being added must all be greater than 6 (otherwise the sum will be less than 30).

   By the assumption of the existence of less than three overlaps, we can write the sum of the four consecutive terms as $S_4 = [6+kd]+[6+(k+1)d]+[6+(k+2)d]+[6+(k+3)d] = 24 + 4kd + 6d$ for some $k \geq 1$. But since $d \geq 1$, we can see that there is no solution for $S_4 = 30$; we have a contradiction.

5. The trick is the add $f(\frac{n}{2015})$ and $f(\frac{2015-n}{2015})$, for each $1 \leq n \leq 2014$. Since

$$f\left(\frac{n}{2015}\right) + f\left(\frac{2015-n}{2015}\right) = \frac{4^{\frac{n}{2015}}}{4^{\frac{n}{2015}} + 2} + \frac{4^{\frac{2015-n}{2015}}}{4^{\frac{2015-n}{2015}} + 2}$$

$$= \frac{4^{\frac{n}{2015}} \cdot 4^{\frac{n}{2015}} \cdot 2}{2 \cdot 4^{\frac{2015-n}{2015}} \cdot 4^{\frac{n}{2015}}} + \frac{4^{\frac{2015-n}{2015}} \cdot 4^{\frac{2015-n}{2015}} \cdot 2}{2 \cdot 4^{\frac{n}{2015}} \cdot 4^{\frac{2015-n}{2015}}}$$

$$= 1.$$

   Hence, $f(1/2015) + f(2/2015) + \cdots + f(2014/2015) = 2014/2 = 1012$.

6. One way to solve this is by using similar triangles, but a much cleaner way to do this is by introducing coordinates. Since the diagonals of $ABCD$ meets at right angles, if we set $(0, 0)$ be the point where the diagonals intercept, then we can assign the coordinates

1. Without loss of generality, suppose . Since $f'(x) = 3x^2 + 2x \quad 5$, the turning points of $f(x)$ are $x = 1$ and $\quad 1\frac{2}{3}$. Furthermore, $f''(x) = 6x + 2$ so $x = 1$ is a minimum and $x = \quad 1\frac{2}{3}$ is a maximum. Therefore, the three roots satis es $\quad < \quad 1\frac{2}{3}$, $\quad 1\frac{2}{3} < \quad < 1$ and $1 <$