MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 2, May 5, 2015 [1]

1.

$$
\begin{aligned}
\frac{y+x}{\sqrt{y}+1} &= x \\
y + x &= (\sqrt{y}+1)\,x \\
y &= (\sqrt{y})x \\
\sqrt{y} &= x
\end{aligned}
$$

2. If $x - y + 2z$ is divisible by 11, then there is an integer k such that $x - y + 2z = 11k$. So we can write

$$
\begin{aligned}
-12x + y - 13z &= -x + y - 2z - 11x - 11z \\
&= -11k - 11(x+z) \\
&= -11(k + x + z).
\end{aligned}
$$

The right hand side of the above equation is divisible by 11, because $k + x + z$ is an integer; $-12x + y - 13z$ is divisible by 11.

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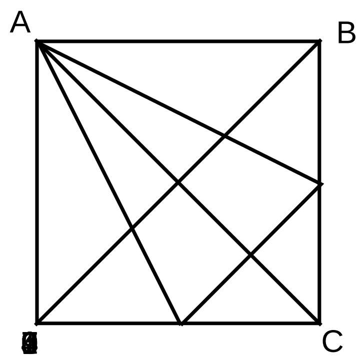
spend moving is $\frac{1}{V_A}\left(\frac{xV_A}{V_A+V_B}+2\right)$. Similar Boris spend $\frac{1}{V_B}\left(\frac{xV_A}{V_A+V_B}-2\right)$ moving. But we already know the difference in timing is $\frac{1}{2}$ hour, therefore

$$\frac{1}{V_A}\left(\frac{xV_A}{V_A+V_B}+2\right)-\frac{1}{V_B}\left(\frac{xV_B}{V_A+V_B}-2\right)=\frac{1}{2};$$

or simplifying to get $\frac{1}{V_A}+\frac{1}{V_B}=\frac{1}{4}$. This expression is symmetric, so if we switch the starting time condition between Anna and Boris, then Anna would cover 2km less and Boris 2km more; $d=2$.

4. Using the notation $|\ |$ to mean the perimeter of a triangle or length of a line. Let the point of intersection between the diagonals DB and AC be O.



(a) Since $ABDC$ is a square, the diagonal BD bisects $\angle ABC$ so that $\angle XBP = 45^\circ$, also the diagonals BD and AC intersections at right angles so that $\angle BOC = 90^\circ$. Furthermore, $\angle COB = \angle PXB$ because XP and CO are parallel. So we have $\angle PXB = \angle BOC = 90^\circ$, which implies $\angle BPX = 180^\circ - 90^\circ - 45^\circ = 45^\circ = \angle XBP$; the triangle PXB is isosceles with $|XB| = |XP|$.

(b) Let $|AB| = a$ and $|OB| = b$, then the perimeter of ABD is $2a+2b$, so we want to show that the perimeter of $|APQ| \le 2a+2b$. Draw a line to connect the points A and X, and let the intersection of PQ and AC be N.

We are given that $|PC| = |QC|$, from this we can work out that the triangles APN and AQN are similar. This implies that N is the midpoint of PQ and $|AP| = |AQ|$. Hence, $|APQ| = 2|NP| + 2|AP|$. Furthermore, $OXPN$ is a parallelogram, thus $|NP| = |OX|$ which implies

$$\begin{aligned}|APQ| &= 2|NP| + 2|AP|\\ &= 2|OX| + 2|AP|\\ &= 2b - 2|XB| + 2|AP|.\end{aligned}$$

Next we work with the side AP to get an upper bound for it. To do this, consider the triangle AXP. The sum of two sides of any triangle is greater than the other side (you may want to think about why this is always true), so $|AP| \le |XP| + |AX| = |XB| + |AX|$, where the second equality on this expression is due to PXB being is isosceles. Also, $|AX| \leq |AB| = a$ because the side AB is opposite to the largest angle in the triangle ABX, from all of this we conclude:

$$
\begin{aligned}
|APQ| &= 2b - 2|XB| + 2|AP| \\
&\leq 2b - 2|XB| + 2|XB| + 2|AX| \\
&\leq 2b + 2a
\end{aligned}
$$

5. Let x be the four digit number we are trying to find. Then $x^2 - x = x(x-1)$ is a number ending in 0000; that is $x(x-1)$ is divisible by $10000 = 2^4 5^4$

The cell with greatest value is $5+x+y=9$, hence $x+y=4$. Also $x \neq y$, otherwise the cells $5+y$ and $5+x$ would have the same number in them; Finally $x, y > 0$ to avoid the cells $x+5$ and $x-5$ being the same.

Because $x \neq y$, we can assume without loss that $x < y$, and since $x+y=4$, we conclude that $x=1$ and $y=3$. Substituting these values into the grid above we obtain the solution given in the problem and hence prove that this solution is unique.

Senior Questions

There was a few typos in the first two equations...

1. Write

$$\begin{aligned} P_3(x) &= a_0 + a_1x^1 + a_2x^2 + a_3x^3 \\ Q_2(x) &= b_0 + b_1x^1 + b_2x^2 \\ R_3(x) &= c_0 + c_1x^1 + c_2x^2 + c_3x^3. \end{aligned}$$

   (a) $P_3(x) \cdot Q_2(x) = a_0b_0 + (a_0b_1 + a_1b_0)x^1 + \dots + (a_3b_2)x^5$, so

   $$P_3(x) \cdot Q_2(x) + R_3(x) = (a_0b_0 + c_0) + (a_0b_1 + a_1b_0)x^1 + \dots + (a_3b_2)x^5 + c_3x^6,$$

   which is a polynomial of degree 6.

   (b) The question should be $P_3(Q_2(\sqrt{x}))$, then

   $$\begin{aligned} P_3(Q_2(\sqrt{x})) &= P_3(b_0 + b_1\sqrt{x} + b_2x) \\ &= a_0 + a_1(b_0 + b_1\sqrt{x} + b_2x)^1 + \dots + a_3(b_0 + b_1\sqrt{x} + b_2x)^3 \end{aligned}$$

   which has degree 3.

2. The equality should be

$$f^{(k)}(x) = P_{2k}\left(\frac{1}{x}\right)\exp\left(\frac{1}{x}\right),$$

which holds for all $k \geq 1$. Note that by definition, $P_k(x)$ means a polynomial of x of degree k, what the real numbers $a_0, \dots, a_k$ are is unimportant for this equation. For $k=1$,

$$\begin{aligned} f^{(1)}(x) &= \frac{d}{dx}\exp\left(\frac{1}{x}\right) \\ &= -\frac{1}{x^2}\exp\left(\frac{1}{x}\right) = P_2\left(\frac{1}{x}\right)\exp\left(\frac{1}{x}\right). \end{aligned}$$

Assuming the expression holds for k, then we want to show that

$$f^{(k+1)}(x) = P_{2(k+1)}\left(\frac{1}{x}\right)\exp\left(\frac{1}{x}\right).$$