MATHEMATICS ENRICHMENT CLUB.

Solution Sheet 12, August 8, 2016

1. Since $1 + 2 + 3 + \cdots + 99 = 4950$ (sum of an arithmetic series), we have

$$n + 2n + 3n + \cdots + 99n = n(1 + 2 + 3 + \cdots + 99) = 4950n.$$

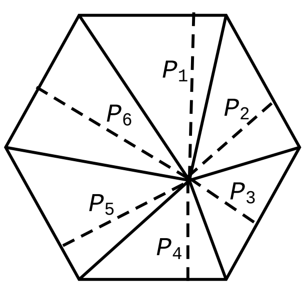
Furthermore, we can factor $4950 = 5^2 \times 3^2 \times 2 \times 11$. Therefore, for $4950n$ to be a perfect square, n must be positive and a multiple of both 11 and 2; the smallest possible n is 22.

2. Since $1 + 2 + 3 + \cdots + n$ is the sum of an arithmetic series, with common difference 1, we have

$$f(n) = \frac{\frac{n}{2}(2 + (n - 1))}{n} = 1 + \frac{1}{2}(n - 1).$$

Therefore $f(1) + f(2) + \cdots + f(100)$ is sum of an arithmetic series, with initial term 1 and common difference 0.5. Hence

$$f(1) + f(2) + \cdots + f(100) = \frac{100}{2}(2 + 99 \times 0.5) = 2575.$$



3. Let $P_1, P_2, \ldots, P_n$ be the sides of the polygon, where n is some nonnegative integer. Then for $1 \le i \le n$, each P_i is the height of a triangle form by the point P and two adjacent corners of the polygon; for example, as shown in the figure above for the case $n = 6$. In particular, the area of the polygon [illegible] of the position of P. Therefore, we conclude by (1) that $P_1 + P_2 + \cdots + P_n$ is the same for all positions of P.

4. Multiplying both sides of the given equation by $(x-2)(x-3)(x-5)$ gives

$$x^2 - p = A(x-3)(x-5) + B(x-2)(x-5) + C(x-2)(x-3). \qquad (2)$$

If we substitute $x = 2$ into (2), then $4 - p = 3A$. Similarly, substituting $x = 3, 5$ into (2) yields $9 - p = -2B$ and $25 - p = 6C$. Therefore, we have the system of equations

$$p = 4 - 3A$$
$$p = 9 + 2B$$
$$p = 25 - 6C.$$

The smallest possible p is 7, with $A = B = -1$ and $C = 3$.

5. We call any 1 cm sides of the lattice an edge, and any point of intersection between edges a vertex. We denote the degree of a vertex by the number of edges incident on that vertex. Consider a vertex with degree 3: since there is an odd number of edges attached to this vertex. Thus, if we can not have multiple threads on an edge, then at least one end of a thread must start at this vertex. In particular, for our 4 cm by 4 cm square lattice, there are 12 vertices with degree 3. Hence, if we only have a total of 40 cm of threads (so that threads can not pass over an edge more than once), then there must be at least $12/2 = 6$ pieces of thread to fill out the lattice.

   (a) No possible, too few threads.

   (b) It is possible.

6. Since $(x + y)^2 = x^2 + 2xy + y$

is 36.

By similar constructions, we can find the last two digits of 3^{2016}, which is 21. Hence, the required number is 57.

Senior Questions

1. $a = 456$, $b = 546$ and $c = 1554$.
2. if $5a^2 - 7b^2 = 9$ then 5 does not divide b hence the remainder on dividing b by 5 is $1; 2; 3$ or 4; i.e $b = 5c + d$, $d = 1; 2; 3$ or 4. Therefore

   $$b^2 = (25c^2 + 10cd) + d^2;$$

   with $d^2 = 1; 4; 9$ or 16. Hence

   $$9 = 5a^2 - 7b^2 = 5(a^2 - 35c^2 - 14cd) - e;$$

   where $e = 7; 28; 63$ or 112. In particular, $5(a^2 - 35c^2 - 14cd)$ is equal to $16; 27; 72$ or 121, which is impossible. Thus, no such b exists.
3. A convex 8 sided polygon with all angles equal has angles $180 - \frac{360}{8} = 135^\circ$. Hence