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MATHEMATICS ENRICHMENT CLUB.

Solution Sheet 4, May 22, 2016

1. Let $x = \sqrt[3]{6 + \sqrt[3]{6 + \sqrt[3]{6 + \sqrt[3]{6 + \cdots}}}}$, then

$$x^3 = 6 + \sqrt[3]{6 + \sqrt[3]{\ }}$$

which implies

$$\begin{aligned} 2\sqrt{(3-x)(x+1)} &< 3 \\ 4(3-x)(x+1) &< 9 \\ -4x^2 + 8x + 3 &< 0. \end{aligned}$$

Solving the above quadratic inequality and recalling that $-1 \le x \le 3$ gives $1 \le x < 1 + \frac{1}{2}\sqrt{7}$.

4. Consider $g(x) := f(x) - 1$. We have

$$g(xy) = f(xy) - 1 = yf(x) + xf(y) - x - y = yg(x) + xg(y).$$

Next, set $h(x) = \frac{g(x)}{x}$ then

$$h(xy) = \frac{g(xy)}{xy} = \frac{g(x)}{x} + \frac{g(y)}{y} = h(x) + h(y).$$

Hence, $h(x) = \ln(x)$. Thus, $f(x) = x\ln(x) + 1$.

5. (a) Let x and y be the two natural numbers. We wish to find the largest value of $xy = x(2016 - x)$, the concaved down parabola with roots 0 and 2016. The maximum is at the turning point $x = 1008$. So the greatest product is $xy = 1008^2$.

   (b) Let $x_1 \le x_2 \le \dots \le x_n$ be the n natural numbers. If any of the natural numbers is 1, then $x_1 = 1$ and the product $x_1x_2\dots x_n$ is not maximum, because we can add x_1 to any one of the other natural numbers and always end up with a greater product.

   Now, suppose that one of natural number say x_1 is greater than 3, then we can split x_1 into $x_1 - 2$ and 2. The result of the product is

$$2 \times (x_1 - 2) \times x_2x_3x_4\dots x_n \ge x_1x_2\dots x_n.$$

   Therefore, the product $x_1x_2\dots x_n$ is greatest when each $x_1, x_2, \dots x_n$ either 2 or 3.

   Finally, if three or more of the natural number are 2's, then we can combine them into two 3's, then product will be greater than it was before since $2 \times 2 \times 2 < 3 \times 3$. Hence, the to obtain the greatest product of $x_1, x_2, \dots x_n$ when $x_1 = x_2 = \dots = x_n = 3$. In particular, the greatest product of the natural numbers that sum to 2016 is $3^{2016/3}$.

6. The sum of angles of a polygon is $(n - 2) \times 180$ (you may want to show this). Hence, the internal angles of each pentagon and decagon is 108° and 144° respectively. Since the length of the sides of both shapes are 1, to make sure there is no gap between tiles, we must join two corners of the a pentagon to each one corner of the decagon to make $2 \times 108^\circ + 144^\circ = 360^\circ$; This will not work without overlaps.

Senior Questions

1. Let $x = 2^a$