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MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 10, 6 August, 2018

1. Let S be the number of members that play Soccer.

   (a) If we add the number of members that play either Basketball, Cricket or Soccer, we end up with a number that is greater than the total number of members in the sports club, because we have double counted the number of members that plays two sports *only*, and triple counted the number of members that plays *all* three.

   So to balance this out we need to subtract the double/triple counts: We know that 10 members play *all* three sports, so these members we triple counted. There are 60 members that plays two or more sports, and 10 that plays all three, therefore there are $60 - 10 = 50$ members that plays two sports *only*.

   The balanced equation is then

   $$163 = S + 100 + 73 - 50 - 2(10),$$

   which gives $S = 60$.

   (b) The number of members that play both Basketball and Cricket but not Soccer is $25 - 10 = 15$, therefore $60 - 15 = 45$ members plays Soccer and Basketball or Soccer and Cricket or all three sports. Since $S = 60$, $60 - 45 = 15$ of these members play Soccer only.

2. (a) Let a_1, k $a_k + 10^{k-1}a_{k-1} + \dots + 10^2 a_3 + 10a_2 + a_1.$

Since 10^i is divisible by 4 for $i = 2, 3, \dots, k$, if n is divisible by 4, then so is $10a_2 + a_1$, which is the number formed by the last two digits of n.

(b) Let m be the number formed by the sum of all of the digits of n; that is

$$m = a_k + a_{k-1} + \dots + a_2 + a_1.$$

Consider the difference

$$n - m = (10^k - 1)a_k + (10^{k-1})a_{k-1} + \dots + 99a_3 + 9a_2.$$

Clearly $n - m$ is a multiple of 9, so if n is divisible by 9, then so is m.

3. We can write the finite sum as

$$1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{10100} = 1 + \sum_{n=2}^{101} \frac{1}{n(n-1)}$$

Using the given formula,

$$\begin{aligned} 1 + \sum_{n=2}^{101} \frac{1}{n(n-1)} &= 1 + \sum_{n=2}^{101} \left(\frac{n-1}{n} - \frac{n-2}{n-1} \right) \\ &= 1 + \sum_{n=2}^{101} \frac{n-1}{n} - \sum_{n=1}^{100} \frac{n-1}{n} \\ &= 1 + \frac{100}{101}. \end{aligned}$$

4. Let the number we wish to express as a continued fraction be n. As given in the hint, $a_0 = \lfloor n \rfloor$, which is easy to calculate. Then

$$n - a_0 = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}},$$

and taking reciprocals, we have

$$\frac{1}{n - a_0} = a_1 + \frac{1}{1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}.$$

Once again, we can see that a_1 is the integer part of $\frac{1}{n - a_0}$. By iteratively taking reciprocals and finding integer parts, we can determine the values of $a_1, a_2, a_3, \ldots$. You should obtain the following results:

(a) $\frac{355}{113} = [3; 7, 16]$

(b) $\frac{113}{355} = [0; 3, 7, 16]$

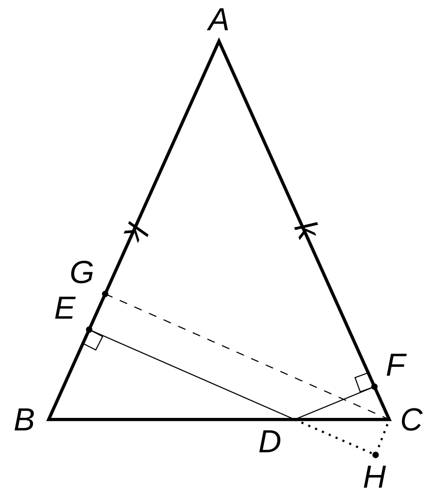
(c) $\sqrt{2} = [1; 2, 2, 2, \ldots]$

(d) $\frac{1}{\sqrt{2}} = [0; 1, 2, 2, 2, \ldots]$

You should notice that (a) and (b) have terminating continued fractions, whereas (c) and (d) have repeating infinite continued fractions. This is, in fact, generally true: rational numbers have terminating continued fractions and quadratic irrationals have infinite continued fractions that repeat.

Also note that (a) and (b) are reciprocals (as are (c) and (d)), and their continued fractions are closely related.

5. (a) Let H be the reflection of F in the line BC. Then DH =You shou Td 230.236 0 Td8 0 Tdd [8



Now $\backslash ABC = \backslash ACB$, since $4ABC$ is isosceles, and since $4EBD$ and