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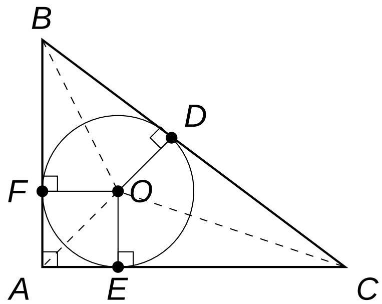
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 7, June 25, 2019[1]

1. Note that x has greater magnitude than y. Firstly, let's concentrate on positive solutions to the equation. If $2019 = x^2 - y^2$, then $2019 = (x + y)(x - y)$. The factors of 2019 are 1, 3, 673, and 2019. So

$(x - y)$	$(x + y)$	x	y
1	2019	1010	1009
3	673	338	335

Thus the solutions are $(1010; \pm 1009)$, $(-1010; \pm 1009)$, $(338; \pm 335)$ and $(-338; \pm 335)$, so there are eight solutions altogether.

2. Let O be the centre of the incircle, let the radius of the incircle be r and let D, E and F be the points of tangency between the incircle and the triangle as shown below.



Since OD, OE, and OF are radii to tangents, $\angle BFO = \angle CEO = \angle ODB = 90^\circ$. Thus $AFOE$ is a square with side length r. Hence $AE = AF = r$, $EC = b - r$ and $FB = c - r$. Furthermore, by RHS, $\triangle EOC \equiv \triangle DOC$ and thus $DC = b - r$. Similarly, $BD = c - r$. Thus

$$a = (b - r) + (c - r)$$
$$r = \frac{1}{2}(b + c - a)$$

[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

3. A neat trick is to express N as

$$\underbrace{333\cdots333}_{61\ 3's} = \frac{3}{9}\left(\underbrace{999\cdots999}_{61\ 9's}\right) = \frac{1}{3}(10^{61}-1).$$

Similarly, $M = \underbrace{666\cdots666}_{62\ 6's} = \frac{2}{3}(10^{62}-1)$. Now

$$\begin{aligned} N\times M &= \frac{2}{9}(10^{61}-1)(10^{62}-1) \\ &= \frac{2}{9}(10^{61}-1)\times 10^{62} - \frac{2}{9}(10^{61}-1) \\ &= \underbrace{222\cdots222}_{60\ 2's}\underbrace{000\cdots000}_{62\ 0's} - \underbrace{222\cdots222}_{60\ 2's} \\ &= \underbrace{222\cdots222}_{60\ 2's}19\underbrace{777\cdots777}_{60\ 7's}8. \end{aligned}$$

4. In modular arithmetic, if $a \equiv b \bmod (n)$, then $a^x \equiv b^x \bmod (n)$. Thus we can see that

$$\begin{aligned} a &\equiv 1 \bmod (a-1) \\ a^x &\equiv 1^x \bmod (a-1) \\ &\equiv 1 \bmod (a-1) \end{aligned}$$

Similarly,

$$\begin{aligned} a &\equiv a \bmod (a+1) \\ &\equiv (-1) \bmod (a+1) \\ a^x &\equiv (-1)^x \bmod (a+1) \end{aligned}$$

$$(-1)^x \bmod (a+1) \equiv \begin{cases} 1 \bmod (a+1) \\ a \end{cases}$$

Thus $r_1 + r_2 = a + 1$.

5. We can write $x = n + d$, where n is the integral part of x and d the decimal part. Then $[2x] + [4x] + [6x] + [8x] = 20n + [2d] + [4d] + [6d] + [8d]$. We scan over the range of d; that is $0 < d < 1$ to see what positive integer under 1001 can be expressed in the form of $[2x] + [4x] + [6x] + [8x]$. For example

$[2x]$	+	$[4x]$	+	$[6x]$	+	$[8x]$		
0	+	0	+	0	+	1	$= 1;$	if $\frac{1}{8} \le d < \frac{1}{6}.$
0	+	0	+	1	+	1	$= 2;$	if $\frac{1}{6} \le d < \frac{1}{4}.$
0	+	1	+	1	+	2	$= 4;$	if $\frac{1}{4} \le d < \frac{1}{3}.$
0	+	1	+	2	+	2	$= 5;$	if $\frac{1}{3} \le d < \frac{3}{8}.$
0	+	1	+	2	+	3	$= 6;$	if $\frac{3}{8} \le d < \frac{1}{2}.$

If we continue with the above calculations, the results are the numbers ending in $3; 7; 8$ or 9 can not be expressed in the form $[2x] + [4x] + [6x] + [8x]$. This means that, for $n = 0$, we have 0 (in this case we can't actually count this one, as we are looking at positive integers), 1, 2, 4, 5 and 6. For $n = 1$, we have 20, 21, 22, 24, 25 and 26 (6 possibilities). For $n = 2$, we have 40, 41, 42, 24, 45 and 46, and so on. Since we are also counting 1000 itself, there are a total of 300 numbers that can be written this way.

6. Let d be the number of kilometres travelled before the tyre switch is made. Then $\frac{d}{x}$ is the proportion of wear on the front tyre before the switch, hence they will travel a further $1 \quad \frac{d}{x} \quad y$ kilometres before the tyres are retired. So the total distance travelled by the font tyre is $d + \quad 1 \quad \frac{d}{x} \quad y$. Similarly, the total distance travelled by the rear tyre is $d + \quad 1 \quad \frac{d}{y} \quad x$.

   Suppose the claim of the advertisement is true, then we must have the following system of inequalities

$$\begin{array}{l} d + \quad 1 \quad \frac{d}{x} \quad y \qquad \frac{x+y}{2} \\ d + \quad 1 \quad \frac{d}{y} \quad x \qquad \frac{x+y}{2} \end{array} :$$

   Rearranging this gives

$$\begin{array}{l} d \;\; 1 \quad \frac{y}{x} \qquad \frac{x \quad y}{2} \\ d \;\; 1 \quad \frac{x}{y} \qquad \frac{y \quad x}{2}; \end{array}$$

   then using the assumption that $x < y$, we have

$$d \quad \frac{x \quad y}{2} \qquad 1 \quad \frac{y}{x}$$

Senior Questions

1. Since $\quad > 0$, $\left(\quad + \frac{1}{\quad} \right)^2 = \quad^2 + \frac{1}{\quad^2} + 2 \quad 2$. Similarly, $\left(\quad + \frac{1}{\quad} \right)^2 \quad 2$. Therefore, if r_1 and r_2 are the roots of f (assuming $r_1 \quad r_2$ wlog), then $r_1 \quad 2$ and $r_2 < 0$, so that $r_1 r_2 = c \quad 3 < 0$, which implies $c < 3$.

   To get the lower bound on c, we use the quadratic formula $2 \quad r_1 = (c+1) + \sqrt{(c+1)^2 \quad 4(c \quad 3)}$. Solving gives $\quad 2 \quad c$.

2. Square both sides of the equation $\sqrt{a} \quad b = \sqrt{c}$ and rearranging gives

$$\sqrt{c} = \frac{a \quad b^2 \quad c}{2b}.$$

   Since the RHS of the above equation is rational, $\sqrt{c}$ must be rational. Write $\sqrt{c} = x/y$, where x and y are integers with greatest common multiplier one. Then $c = x^2/y^2$, and greatest common multiplier between x^2 and y^2 is one. Since c is an integer, x^2 must be divisible by y^2, which can only happen if $y^2 = 1$, because the greatest common multiplier between x^2 and y^2 is one. Hence $c = x^2$, so that c is a perfect square.

   If c is a perfect square, then the equation $\sqrt{a} \quad b = \sqrt{c}$ implies that a is also a perfect square.

3. Use the method of reflection. Reflect the point B in the line that represents the river bank. This is shown as B' in the diagram below. Then the shortest distance from A to B' is clearly a straight line. We can use Pythagoras' theorem to show that this is 15 km.