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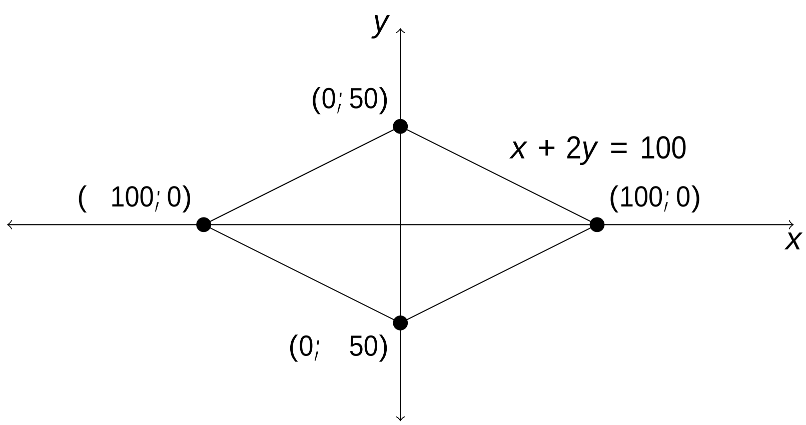
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 8, July 1, 2019

1. First note that x must be an even number, so let's consider the possible values of y.
   If $y = 0$, then $x =$ 100 (2 solutions).
   If $y =$ 1, then $x =$ 98 (4 solutions)
   If $y =$ 2, then $x =$ 96 (4 solutions)
   However, if $y =$ 50, then $x = 0$ (2 solutions again).
   So there are 49 4 + 2 2 = 200 solutions.

   The following graphical solution was contributed by a student.



(b) $y^3 - 5\{y\} = 10$ so $10 < y^3 < 15$. Thus $2 < y < 3$, $y = 2 + \{y\}$ and so $\{y\} = y - 2$. Hence $y^3 = 10 + 5(y - 2) = 5y$, and since $y \neq 0$, $y^2 = 5$ and hence $y = \sqrt{5}$.

4. There are two possibilities: either $f(x)$ is the product of a linear polynomial and a cubic or two quadratics. In the first case, this means that, for some integers a, b, c and d,

$$\begin{aligned} x^4 - nx + 63 &= (x + a)(x^3 + bx^2 + cx + d) \\ &= x^4 + (a + b)x^3 + (ab + c)x^2 + (ac + d)x + ad \end{aligned}$$

Equating coefficients, we have

$$\begin{aligned} a + b &= 0 && (1) \\ ab + c &= 0 && (2) \\ ac + d &= -n && (3) \\ ad &= 63 && (4) \end{aligned}$$

From (1), we have $b = -a$, which substituted into (2) gives $c = a^2$. If we substitute this into (3), we have $n = -(a^3 + d)$. Thus all the coefficients can be written in terms of a and d alone. Since $ad = 63$, both a and d have the same sign. We will consider them both negative, then the sign of n is positive and we can draw up the following table:

a	d	$n = -(a^3 + d)$
-1	-63	64
-3	-21	48
-7	-9	352
-9	-7	736
-21	-3	9264
-63	-1	250 048

In this case, the smallest value of n is 48.

Now let's consider the two quadratics case. Then

$$\begin{aligned} x^4 - nx + 63 &= (x^2 + ax + b)(x^2 + cx + d) \\ &= x^4 + (a + c)x^3 + (b + d + ac)x^2 + (bc + ad)x + bd \end{aligned}$$

Equating coefficients, we have

$$\begin{aligned} a + c &= 0 && (1) \\ b + d + ac &= 0 && (2) \\ bc + ad &= -n && (3) \\ bd &= 63 && (4) \end{aligned}$$

From (1), we have $a = \quad c$, which substituted into (2) gives $b + d = c^2$; and substituted into (3) gives

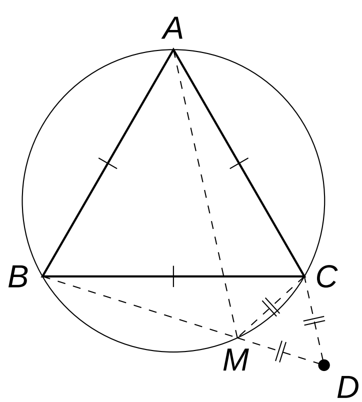
$$bc \quad cd = \quad n$$
$$c(b \quad d) = \quad n$$
$$n = c(d \quad b)$$

Thus we have

b	d	$c^2 = b + d$	c	$n = c(d \quad b)$
1	63	64	8	496
3	21	24	Not valid	
7	9	16	4	8

So the smallest value of n is 8.

5. Extend the line BM to the point D where $DM = CM$. Then $BD = MB + MC$. Since $ACMB$ is a cyclic quadrilateral and $4ABC$ is equilateral, $\backslash CMD = \backslash BAC = 60$. So $4CMD$ is also equilateral. It can be shown by SAS that $4ACM \quad 4BCD$, and hence $AM = BD = MD + MC$.



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3. Since $2n+1$ is odd and a perfect square, we can write as $2n+1 = (2k+1)^2 = 4k^2+4k+1$, for some non-negative integer k, which implies $n = 2k(k+1)$. Since either k or $k+1$ is odd, we conclude that n is a multiple of 4.

   Because n is even, $3n+1$ must be odd so we can write $3n+1 = (2j+1)^2$, for some non-negative j, which implies $3n = 4j(j+1)$. Similar to before, either j or $j+1$ is odd, so we can conclude that n is divisible by 8.

   To complete the question, we show that n is divisible by 5. The possible remainder of an integer a divided by 5 are $0; 1; 2; 3$ and 4, therefore any perfect square number must have remainders $0; 1^2; 2^2; 3^2 \quad 5$ and $4^2 \quad 3(5)$; that is $0; 1; 4$ are the only remainders of a perfect square number when divided by 5. If we consider the remainders of $2n+1$ and which 1; 2; 2; 24.446 T.446 T.t[(+g4 5.)-433(T)9F1764which im 418videdrf10.055 0 Td [(or)]T